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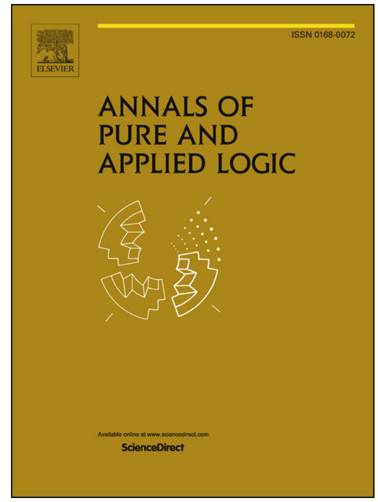

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Andrés Cordón-Franco, F. Félix Lara-Martín, Manuel J.S. Loureiro

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Lipschitz and Wadge binary games in second order arithmetic

Andrés Cordón-Franco, F.Félix Lara-Martín

Dpto. Ciencias de la Computación e Inteligencia Artificial
Facultad de Matemáticas. Universidad de Sevilla
C/ Tarfia, s/n, 41012 Sevilla (Spain)

Manuel J. S. Loureiro

Faculty of Engineering, Lusofona University,
Campo Grande 376, 1749-024 Lisbon (Portugal)

Abstract

We present a detailed formalization of Lipschitz and Wadge games in the context of second order arithmetic and we investigate the logical strength of Lipschitz and Wadge determinacy, and the tightly related Semi-Linear Ordering principle, for the first levels of the Hausdorff difference hierarchy in the Cantor space. As a result, we obtain characterizations of WKL_0 and ACA_0 in terms of these determinacy principles.

Keywords: Reverse mathematics, Determinacy, Wadge games, Semilinear Ordering principle
2020 MSC: 03B30, 03E60, 03F35

1. Introduction

Lipschitz and Wadge games were first introduced in descriptive set theory in the late 1960's by W.W. Wadge (see [17]) as a tool for studying the relative complexity of subsets of the Baire space ω^ω. Given subsets A and B, A is said to be Wadge reducible to B, in symbols $A \leq_W B$, if there is a continuous function F such that $x \in A$ if, and only if, $F(x) \in B$ (the problem of verifying membership in A can be reduced to the problem of verifying membership in B and so A is, in a certain sense, no more complicated than B.) In a similar way, A is said to be Lipschitz reducible to B, in symbols $A \leq_L B$, if the previous function F is required to be a Lipschitz function. Wadge proved that the reducibility relations $\leq_W$ and $\leq_L$ can be naturally studied in terms of the so-called *Wadge and Lipschitz games*. These are two person infinite games with perfect information in which each player has a different pay-off set A and B, respectively, and player II wins the game if she mimics player I's resulting play, that is, if she plays inside B if and only if player I plays inside A (see section 2 for a precise definition of these games). Wadge then assumed determinacy for Wadge and Lipschitz games as a working hypothesis and he extensively studied the structure of the equivalence classes generated by $\leq_W$ and $\leq_L$ in the Baire space, especially of those equivalence classes formed by Borel sets. In particular, he derived the following somewhat surprising comparability property, known as the *Semi-Linear Ordering principle*:

Email addresses: `acordon@us.es` (Andrés Cordón-Franco), `fflara@us.es` (F.Félix Lara-Martín), `mloureiro@ulusofona.pt` (Manuel J. S. Loureiro)

(SLO) = "For all subsets A and B, either A is reducible to B or the complement of B is reducible to A."

Since then the use of Wadge/Lipschitz games has been shown to be a useful tool in descriptive set theory and many authors have contributed to this line of research.

The reverse mathematics of the determinacy of general two person infinite games has been thoroughly investigated by, among others, J.R. Steel, K. Tanaka, M.O. MedSalem, and T. Nemoto ([14], [15], [16], [11], [9]). As a result, we now have a detailed level-by-level analysis of the logical strength of these determinacy principles both in the Baire space ω^ω and in the Cantor space 2^ω and, in most cases, their exact strength has been calibrated in terms of subsystems of second order arithmetic. In contrast, the situation for Lipschitz and Wadge determinacy is completely different. By a remarkable result of A. Louveau and J. Saint Raymond [6] we know that full second order arithmetic Z_2 proves Borel Wadge/Lipschitz determinacy, but no detailed analysis of the strength of Lipschitz or Wadge determinacy in terms of subsystems of second order arithmetic can be found in the literature and, to the best of our knowledge, the reverse mathematics of the semi-linear ordering principle SLO has not been systematically investigated either. A first step towards filling this gap was M.J.S. Loureiro's Ph.D. thesis [5], where the author systematically studied the reverse mathematics of Wadge/Lipschitz determinacy and SLO for the first levels of the Borel hierarchy both in the Baire space and in the Cantor space. In [5] the author was able to characterize the central second arithmetic theories ACA_0 and ATR_0 using these determinacy principles, and he raised the question of characterizing the system WKL_0 in terms of Lipschitz determinacy in the Cantor space. Let us introduce some terminology in order to state more precisely these results. Given two formula classes Γ_1 and Γ_2 in the language of second order arithmetic, let $(\Gamma_1,\Gamma_2)\text{-}\mathsf{Det}^*_{L/W}$ denote the principle of Lipschitz/Wadge determinacy for games in the Cantor space where player I's pay-off set is Γ_1-definable and player II's pay-off set is Γ_2-definable. Likewise, let $(\Gamma_1,\Gamma_2)\text{-}\mathsf{SLO}^*_{L/W}$ denote the corresponding semi-linear ordering principle in the Cantor space. Finally, we will write $(\Gamma_1,\Gamma_2)\text{-}\mathsf{Det}_{L/W}$ and $(\Gamma_1,\Gamma_2)\text{-}\mathsf{SLO}_{L/W}$ for the corresponding principles in the Baire space (we refer the reader to sections 2 and 5 for a detailed formalization of these principles in the language of second order arithmetic.) The main results of [5] are the following (note that we will simply write $\Gamma\text{-}\mathsf{Det}^{(*)}_{L/W}$ or $\Gamma\text{-}\mathsf{SLO}^{(*)}_{L/W}$ if $\Gamma_1=\Gamma_2=\Gamma$.)

Theorem 1.1. *(Theorem 4.12, Proposition 4.18, Corollary 4.42, Theorem 5.1, Theorem 5.28, Theorem 4.37 and Proposition 5.33 in [5])*

1. RCA_0 *proves* $\Delta^0_1\text{-}\mathsf{Det}^*_W$*, and* WKL_0 *proves* $(\Delta^0_1,\Sigma^0_1)\text{-}\mathsf{Det}^*_{L/W}$ *and* $(\Sigma^0_1,\Delta^0_1)\text{-}\mathsf{Det}^*_L$.
2. *Over* RCA_0, $(\Sigma^0_1\wedge\Pi^0_1)\text{-}\mathsf{Det}^*_L$, $(\Sigma^0_1\wedge\Pi^0_1)\text{-}\mathsf{SLO}^*_L$ *and* ACA_0 *are pairwise equivalent.*
3. *Over* ACA_0, $\Delta^0_1\text{-}\mathsf{Det}_L$, $\Delta^0_1\text{-}\mathsf{SLO}_L$ *and* ATR_0 *are pairwise equivalent.*
4. *Over* RCA_0, $\Sigma^0_1\text{-}\mathsf{Det}_L$, $(\Delta^0_1,\Sigma^0_1)\text{-}\mathsf{Det}_L$ *and* ATR_0 *are pairwise equivalent.*
5. *Let* $\Gamma=\Sigma^0_1\wedge\Pi^0_1$. *Then,* ACA_0 *proves* $\Gamma\text{-}\mathsf{Det}^*_W$ *and* $\Pi^1_1\text{-}\mathsf{CA}_0$ *proves* $(\Gamma\cup\neg\Gamma)\text{-}\mathsf{Det}_{L/W}$.

Also, the following question was left pending in [5]:

Problem 1. (Problem 4.17 in [5]) Is WKL_0 equivalent over RCA_0 to $\Delta^0_1\text{-}\mathsf{Det}^*_L$?

In preparing the final version of the present paper, we have learned that W. Chan, possibly following a suggestion of T. Nemoto, had also considered it interesting to develop a more detailed analysis of Borel Wadge/Lipschitz determinacy in terms of subsystems of second order arithmetic already in 2011-2012, although his work in this direction had remained unpublished. Nevertheless, motivated by the aforementioned question on WKL_0 posed in [5], in his recent work [4] Chan has proved WKL_0 to be equivalent to a certain weak form of Lipschitz determinacy in the Cantor space, thus giving a first partial answer to that question. More precisely,

Theorem 1.2. *([4]) Over* RCA_0, $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{Det}^*_L$ *and* RCA_0 *are pairwise equivalent.*

The current paper builds on the work in [5] and presents extensions and improved versions of the main results in [5] concerning the Cantor space (a second paper dealing with Lipschitz/Wadge determinacy and SLO in the Baire space is in preparation.) Most remarkably, we give a complete answer to Problem 1 above by showing that Δ^0_1-Det^*_L already implies WKL_0 over RCA_0, and we improve the reversals for ACA_0 previously obtained in [5]. The following theorem summarizes the main results we prove in this paper.

Theorem 1.3.

1. RCA_0 *proves* (Δ^0_1, Σ^0_n)-Det^*_W, (Δ^0_1, Σ^0_n)-SLO^*_W *and* (Σ^0_n, Δ^0_1)-SLO^*_W, *for* $n > 0$.
2. *Over* RCA_0, Δ^0_1-Det^*_L, (Δ^0_1, Σ^0_1)-Det^*_L, (Σ^0_1, Δ^0_1)-Det^*_L *and* WKL_0 *are pairwise equivalent.*
3. *Over* WKL_0, Σ^0_1-$\mathsf{Det}^*_{L/W}$, Σ^0_1-$\mathsf{SLO}^*_{L/W}$ *and* ACA_0 *are pairwise equivalent.*
4. *Over* RCA_0, Σ^0_1-Det^*_L, $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*_{L/W}$, $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}^*_{L/W}$ *and* ACA_0 *are pairwise equivalent.*

Some remarks concerning Theorem 1.3 are in order.

Firstly, it is important to note that the implications from WKL_0 and ACA_0 to *Lipschitz* determinacy statements can also be derived from previous general results on determinacy obtained by Nemoto, MedSalem and Tanaka in [11] and [9]. This is because every Lipschitz game can be effectively reduced to a general two person infinite game. Roughly speaking, one can infer Lipschitz determinacy for Γ sets from general determinacy for $(\Gamma \wedge \neg\Gamma)$ sets (see Remark 2.4 for details). This reduction provides us with some upper bounds on the strength of Lipschitz determinacy but, in general, these upper bounds need not be optimal. Actually, this will be the case for higher levels of the Borel hierarchy, as by the aforementioned result of A. Louveau and J. Saint Raymond Z_2 proves Borel Lipschitz determinacy whereas by a result of D. Martin [7], Z_2 does not prove general Σ^0_4-determinacy (this result was improved by A. Montalbán and R.A. Shore [8] showing that Z_2 does not prove general Δ^0_4-determinacy.) This justifies the methodology used in the present paper: we give explicit formalizations of Lipschitz and Wadge games in second order arithmetic, and we give direct proofs of Lipschitz and Wadge determinacy with no use of previously known results on general determinacy. In fact, we show that the topological analysis of the complete sets in the first levels of the Hausdorff difference hierarchy developed in [17] can be adapted to prove the determinacy of Lipschitz/Wadge games of those complexities (recall that in [17] the author was not concerned with proving determinacy results but he assumed determinacy of Lipschitz and Wadge games as an initial hypothesis for his work.) The use of topological arguments makes, we think, our determinacy proofs conceptually simple: very roughly speaking, the player who plays in a pay-off set of richer topological structure will win the game.

Secondly, let us observe that the implications from RCA_0, WKL_0 and ACA_0 to *Wadge* determinacy statements in Theorem 1.3 are new results, in the sense that they do not follow from previous general results on determinacy (Wadge determinacy for Γ sets can be naturally reduced to general determinacy for $max\{\Sigma^0_2, \Gamma \wedge \neg\Gamma\}$ sets, see Remark 2.4) The proofs of these implications can also be found in the Ph.D. thesis [5] (unpublished).

Finally, it should be noted that the reversals for WKL_0 and ACA_0 in Theorem 1.3 are genuinely new results. On the one hand, they cannot be deduced from (and, indeed, can be seen as reinforcements of) previous general results on determinacy obtained in [11] and [9], for Lipschitz/Wadge determinacy and SLO are formally weaker principles than general determinacy. On the other hand, our reversal for the Weak Köning Lemma (namely, Δ^0_1-Det^*_L implies WKL_0 over RCA_0) is stronger than the one obtained in [4], where the author needed $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{Det}^*_L$ to recover WKL_0 over RCA_0. In a similar vein, our reversals for Arithmetical Comprehension (namely, Σ^0_1-Det^*_L and $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}^*_{L/W}$ imply ACA_0 over RCA_0) are stronger than the ones obtained in [5], where $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*_L$ or $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}^*_L$ are needed to recover ACA_0 over RCA_0.

The paper is divided into six sections. Section 1 is introductory. Section 2 contains some preliminaries and describes a detailed formalization of Lipschitz and Wadge games in second order arithmetic. Although in this paper we shall only deal with games in the Cantor space, we have preferred to present this formalization in the more general setting where players I and II play in the space $X^{\mathbb{N}}$ with $X \subseteq \mathbb{N}$, as this could be useful for future references. Sections 3 is devoted to the study of clopen Lipschitz/Wadge determinacy and contains a reversal for WKL_0 in terms of clopen Lipschitz determinacy. In section 4 we show that ACA_0 proves open Lipschitz/Wadge determinacy in the Cantor space, and we prove a combinatorial version of a result of P. Shafer [12] stating that ACA_0 is equivalent to the topological property: "every closed and not open set in the Cantor space has nonempty boundary." Reversals for ACA_0 are postponed until section 5, in which we initiate the study of the reverse mathematics of the semi-linear ordering principle SLO and we establish two reversals for ACA_0 in terms of this principle. Section 6 contains some concluding remarks.

2. Lipschitz and Wadge games in second order arithmetic

We assume familiarity with subtheories of second-order arithmetic, as presented in [13]. Of the "big five" theories thoroughly studied in that book, here we will only deal with the three weakest: $\mathsf{RCA}_0 \subset \mathsf{WKL}_0 \subset \mathsf{ACA}_0$. The theory of *recursive comprehension* RCA_0 is axiomatized (over a finite set of basic axioms) by Δ^0_1 comprehension and Σ^0_1 induction. We will use RCA_0 as our base theory through this paper. The axioms of WKL_0 are those of RCA_0 plus *weak Köning's lemma* stating that every infinite binary tree has a path. The theory ACA_0 extends RCA_0 by *arithmetical comprehension*.

Our notation and terminology are standard and follow [13] (for details and full technical background the reader should consult that book). Here we restrict ourselves to recalling some basic notions concerning functions and finite sequences that will be used extensively in this paper. Within RCA_0, we define $\mathbb{N}$ to be the unique set X such that $\forall n\,(n \in X)$ and we define a numerical paring function by letting $(m, n) = (m + n)^2 + m$. Using Δ^0_1 comprehension, we can prove that for all sets X, $Y \subseteq \mathbb{N}$, there exists a set $X \times Y \subseteq \mathbb{N}$ consisting of all (m, n) such that $m \in X$ and $n \in Y$. A function $f : X \to Y$ is defined to be a set $f \subseteq X \times Y$ such that for all $m \in X$ there is exactly one $n \in Y$ such that $(m, n) \in f$. For $m \in X$, $f(m)$ is defined to be the unique n such that $(m, n) \in f$. We write $f \in X^{\mathbb{N}}$ to mean that f is a function from $\mathbb{N}$ to X, although the set $X^{\mathbb{N}}$ does not formally exist within second order arithmetic. For $X = 2 = \{0, 1\}$ we identify functions in $2^{\mathbb{N}}$ with points in the Cantor space, and for $X = \mathbb{N}$ we identify functions in $\mathbb{N}^{\mathbb{N}}$ with points in the Baire space. Finite sequences of natural numbers can be encoded as a single natural number and this coding can be developed formally within RCA_0. The set of all (codes of) finite sequences from X is denoted $X^{<\mathbb{N}}$. The *empty sequence* is denoted $\langle\rangle$. Given any $s, t \in X^{<\mathbb{N}}$, $|s|$ denotes the length of s, $s(i)$ denotes the $(i+1)$-th element of s for $i < |s|$, and, for each $n \leq |s|$, $s\,[n]$

is the *n-th initial segment of* s, i.e. $\langle s(0), ..., s(n-1)\rangle$. If $s = t[n]$ for some $n \leq |t|$, we write $s \subseteq t$ and say that s is an initial segment of t (or t an extension of s). The concatenation of s and t, written $s * t$, is the sequence $\langle s(0), ..., s(|s|-1), t(0), ..., t(|t|-1)\rangle$. If $f \in X^{\mathbb{N}}$, $s * f$ denotes $\langle s(0), ..., s(|s|-1), f(0), f(1), ..., f(n), ...\rangle$, and $f[n]$ denotes $\langle f(0), ..., f(n-1)\rangle$. If $s = f[|s|]$, we write $s \subset f$ and say that s is an initial segment of f (or f is an extension of s). Note that the relations $s = \langle\rangle$, $|s| = n$, $s(i) = n$, $s \subseteq t$, $s = t * u$, $s = t[n]$, ... are formally defined as Σ^0_0 formulas in RCA_0.

A formalization of *classical* two-person infinite games within second order arithmetic is described in section V.8 of [13] as well as in section 3 of [11]. To fix notation and terminology, we recall some basic notions. Our starting point is a given formula $\varphi(f)$ with a distinguished function variable $f \in X^{\mathbb{N}}$ and possibly other first and second order parameters. The language of second order arithmetic does not formally contain any function variables but one can naturally express the fact that "f is a function from $\mathbb{N}$ to X" by using a Π^0_2 formula. (The price to pay is a possible increase of the quantifier complexity of the formulas involved. However, this is unimportant when working in the Cantor space, i.e. $X = \{0, 1\}$, because in this case one can regard f as a set variable by identifying $f(n) = 0$ and $f(n) = 1$ with $n \notin f$ and $n \in f$, respectively.) An infinite game with *pay-off* set $\varphi(f)$, denoted $G^X(\varphi)$, is defined as follows: Two players, say player I (male) and player II (female), alternately choose an element x in X to form $f \in X^{\mathbb{N}}$ which is called the *resulting play.* Player I plays first. Player I wins if and only if $\varphi(f)$ holds. Otherwise Player II wins that play. If we define $\mathrm{Seq}^X_{even} = \{s \in X^{<\mathbb{N}} : |s| \text{ is even}\}$ and $\mathrm{Seq}^X_{odd} = \{s \in X^{<\mathbb{N}} : |s| \text{ is odd}\}$, then a *strategy* for player I in the game $G^X(\varphi)$ is a function $\sigma_{\mathrm{I}} : \mathrm{Seq}^X_{\mathrm{even}} \to X$ and a strategy for player II in $G^X(\varphi)$ is a function $\sigma_{\mathrm{II}} : \mathrm{Seq}^X_{\mathrm{odd}} \to X$. If players I and II follow strategies σ_{I} and σ_{II}, respectively, the resulting play is uniquely determined and denoted by $\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}$. In fact, $\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}$ is the function $h : \mathbb{N} \to X$ defined by the recursive equations $h(2k) = \sigma_{\mathrm{I}}(h[2k])$ and $h(2k+1) = \sigma_{\mathrm{II}}(h[2k+1])$. A strategy for a player is a *winning strategy* if the player wins the game as long as he/she plays following it, no matter what his/her opponent plays. A game $G^X(\varphi)$ is *determined* if either player I or player II has a winning strategy:

$$\mathsf{Det}^X(\varphi) \equiv \exists\sigma_{\mathrm{I}}\,\forall\sigma_{\mathrm{II}}\,\varphi(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}) \,\vee\, \exists\sigma_{\mathrm{II}}\,\forall\sigma_{\mathrm{I}}\,\neg\varphi(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}),$$

where σ_{I} and σ_{II} range over strategies for players I and II, respectively. Given a prescribed formula class Γ, Γ-determinacy axiom schemes declare that all games of pay-off set in Γ are determined. Thus,

- The scheme of Γ-determinacy in the Baire space, denoted Γ-Det, consists of all axioms $\mathsf{Det}^{\mathbb{N}}(\varphi)$, where $\varphi(f)$ is in Γ.
- The scheme of Γ-determinacy in the Cantor space, denoted Γ-Det^*, consists of all axioms $\mathsf{Det}^{\{0,1\}}(\varphi)$, where $\varphi(f)$ is in Γ.

We are now in a position to describe a detailed formalization of Lipschitz and Wadge games in second order arithmetic as well as to introduce the Lipschitz and Wadge determinacy axiom schemes we will be interested in.

2.1. Formalized Lipschitz games

Fix $X \subseteq \mathbb{N}$ nonempty and consider formulas $A(f)$ and $B(g)$ with distinguished function variables $f, g \in X^{\mathbb{N}}$. A *Lipschitz game in the space* $X^{\mathbb{N}}$, denoted $G^X_L(A, B)$, is defined as follows: Two players, say player I (male) and player II (female), alternately choose an element x in X to form the resulting plays $f = \langle x_0, x_1, x_2, \dots \rangle \in X^{\mathbb{N}}$ and $g = \langle y_0, y_1, y_2, \dots \rangle \in X^{\mathbb{N}}$, respectively.

Player I	x_0		x_1		x_2		$\dots$
Player II		y_0		y_1		y_2	$\dots$

Player I wins if and only if $\neg(A(f) \leftrightarrow B(g))$ holds. Player II wins if and only if $(A(f) \leftrightarrow B(g))$ holds.

A *strategy for player* I in the game $G^X_L(A,B)$ is a function $\sigma_{\mathrm{I}} : \mathrm{Seq}^X_{\mathrm{even}} \to X$ and a *strategy for player* II in the game $G^X_L(A,B)$ is a function $\sigma_{\mathrm{II}} : \mathrm{Seq}^X_{\mathrm{odd}} \to X$. If players I and II follow strategies σ_{I} and σ_{II}, respectively, the resulting plays are uniquely determined. We will write $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}}$ to denote player I's resulting play and write $\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}}$ to denote player II's resulting play. Note that $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}}(n) = \sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2n)$ and $\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}}(n) = \sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2n+1)$.

The following axiom, denoted $\mathsf{Det}^X_L(A,B)$, expresses that the Lipschitz game $G^X_L(A,B)$ is *determined*:

$$\exists \sigma_{\mathrm{I}}\, \forall \sigma_{\mathrm{II}}\, \neg(A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}})) \vee \exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\, (A(\sigma_{\mathrm{I}} \otimes^{I}_L \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}})),$$

where σ_{I} and σ_{II} range over strategies for players I and II, respectively.

Definition 2.1. Fix $X = \{0,1\}$ or $\mathbb{N}$. Let Γ_1 and Γ_2 be classes of formulas with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively.

1. The scheme of (Γ_1, Γ_2)-Lipschitz determinacy in $X^{\mathbb{N}}$, denoted (Γ_1, Γ_2)-Det^X_L, is given by the axioms $\mathsf{Det}^X_L(A,B)$, where $A(f)$ is in Γ_1 and $B(g)$ is in Γ_2. For simplicity, if $\Gamma_1 = \Gamma_2 = \Gamma$, we will write Γ-Det^X_L instead of (Γ, Γ)-Det^X_L.
2. The scheme of (Δ^0_n, Δ^0_m)-Lipschitz determinacy in $X^{\mathbb{N}}$, denoted (Δ^0_n, Δ^0_m)-Det^X_L, is given by the axioms

$$\Big(\forall f \in X^{\mathbb{N}}(A(f) \leftrightarrow C(f)) \wedge \forall g \in X^{\mathbb{N}}(B(g) \leftrightarrow D(g))\Big) \to \mathsf{Det}^X_L(A,B)$$

where $A(f)$ is in Σ^0_n, $C(f)$ is in Π^0_n, $B(g)$ is in Σ^0_m and $D(g)$ is in Π^0_m. For simplicity, if $n = m$, we will write Δ^0_n-Det^X_L instead of (Δ^0_n, Δ^0_n)-Det^X_L.

The schemes (Γ, Δ^0_n)-Det^X_L and (Δ^0_n, Γ)-Det^X_L are defined similarly. We will omit the superscript $\mathbb{N}$ for determinacy schemes in the *Baire space* and write simply (Γ_1, Γ_2)-Det_L, Δ^0_n-Det_L, etc.; whereas following [11], we will replace the superscript $\{0,1\}$ with $*$ and write (Γ_1, Γ_2)-Det^*_L, Δ^0_n-Det^*_L, ... to denote the corresponding determinacy schemes in the *Cantor space*.

2.2. Formalized Wadge games

Fix $X \subseteq \mathbb{N}$ nonempty and consider formulas $A(f)$ and $B(g)$ with distinguished function variables $f, g \in X^{\mathbb{N}}$. A *Wadge game in the space* $X^{\mathbb{N}}$, denoted $G^X_W(A,B)$, is defined as follows: Two players, say player I (male) and player II (female), alternately choose an element x in X to form the resulting plays $f = \langle x_0, x_1, x_2, \dots \rangle \in X^{\mathbb{N}}$ and $g = \langle y_0, y_1, y_2, \dots \rangle \in X^{\mathbb{N}}$, respectively. Player I plays first and in each of his turns he must choose an element x in X. In each of her turns, player II either chooses an element x in X or has the option to pass but she has to play infinitely often otherwise she loses (below $\mathtt{p}$ denotes that player II passes):

Player I	x_0		x_1		x_2		x_3		$\dots$
Player II		y_0		$\mathtt{p}$		$\mathtt{p}$		y_1	$\dots$

Player I wins if player II does not play infinitely often or $\neg(A(f) \leftrightarrow B(g))$ holds. Player II wins if she plays infinitely often and $(A(f) \leftrightarrow B(g))$ holds.

When codifying a run of the play, for player II we will identify picking the number zero with passing and picking the number $x+1$ with choosing $x \in X$ to form her resulting play. As a consequence, player II will actually play in the set $X^+ = \{0\} \cup \{i+1 : i \in X\}$. As for player I, we opt for allowing him to play in set X^+ too and we will identify picking $x \in X^+$ with choosing $x \dot{-} 1 \in X$ to form his resulting play (where $\dot{-}$ denotes the modified subtraction function given by $a \dot{-} b = \max(0, a-b)$.) As an example, a run of a Wadge game codified by $2, 2, 0, 0, 1, 1, 2, 0, \ldots$ will correspond to

Player I	1		0		0		1		...
Player II		1		p		0		p	...

Note that the code of a run of the play is not unique since for player I, choosing 0 to form his resulting play can be codified by using either 0 or 1. (For example, $2, 2, 1, 0, 0, 1, 2, 0, \ldots$ would be a different code of the previous run.) The following bounded formula $\theta(s,t)$ expresses that the finite sequences s and t are two codes of the same partial run of a Wadge game:

$$|s| = |t| \wedge \forall i < |s|\,(2i < |s| \rightarrow s(2i) \dot{-} 1 = t(2i) \dot{-} 1) \wedge \forall i < |s|\,(2i+1 < |s| \rightarrow s(2i+1) = t(2i+1)).$$

A *strategy for player* I in the game $G_W^X(A,B)$ will be a function $\sigma_{\mathrm{I}} : \mathrm{Seq}_{\mathrm{even}}^{X^+} \rightarrow X^+$, and a *strategy for player* II will be a function $\sigma_{\mathrm{II}} : \mathrm{Seq}_{\mathrm{odd}}^{X^+} \rightarrow X^+$. But now we must require strategies to be *coherent*, in the sense that the output of the strategy does not depend on the particular code of a run of the play, that is: $\forall s,t \in \mathrm{Seq}_{\mathrm{even}}^{X^+}\,(\theta(s,t) \rightarrow \sigma_{\mathrm{I}}(s) = \sigma_{\mathrm{I}}(t))$ and $\forall s,t \in \mathrm{Seq}_{\mathrm{odd}}^{X^+}\,(\theta(s,t) \rightarrow \sigma_{\mathrm{II}}(s) = \sigma_{\mathrm{II}}(t))$. From now on, we assume that for Wadge games 'strategy' means 'coherent strategy.' Given strategies σ_{I} and σ_{II}, the restriction that player II must play infinitely often can be expressed by the Π^0_2 formula

$$\mathsf{Inf}(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \equiv \forall n\, \exists k > n\, ((\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})(2k+1) \neq 0).$$

If players I and II follow strategies σ_{I} and σ_{II}, respectively, and $\mathsf{Inf}(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}})$ holds, the resulting plays are uniquely determined. We will write $\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}$ to denote player I's resulting play and write $\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}$ to denote player II's resulting play. Note that both $\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}$ and $\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}$ will be functions from $\mathbb{N}$ into X. Actually, we have $\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}(n) = (\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2n)) \dot{-} 1$ and $\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}(n) = (\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}}(2 \cdot \mathit{move}(n) + 1)) - 1$, where

$$\begin{aligned} \mathit{move}(0) &= \mu i\,[(\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})(2i+1) \neq 0] \\ \mathit{move}(n+1) &= \mu i\,[i > \mathit{move}(n) \wedge (\sigma_{\mathrm{I}} \otimes \sigma_{\mathrm{II}})(2i+1) \neq 0]. \end{aligned}$$

The following axiom, denoted $\mathsf{Det}_W^X(A,B)$, expresses that the Wadge game $G_W^X(A,B)$ is *determined*:

$$\begin{aligned} &\exists \sigma_{\mathrm{I}}\, \forall \sigma_{\mathrm{II}}\, \left[\mathsf{Inf}(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \rightarrow \neg(A(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}))\right] \\ \vee\, &\exists \sigma_{\mathrm{II}}\, \forall \sigma_{\mathrm{I}}\, \left[\mathsf{Inf}(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \wedge (A(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}))\right], \end{aligned}$$

where σ_{I} and σ_{II} range over strategies for players I and II, respectively.

Definition 2.2. Fix $X = \{0,1\}$ or $\mathbb{N}$. Let Γ_1 and Γ_2 be classes of formulas with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively.

1. The scheme of (Γ_1, Γ_2) Wadge determinacy in $X^{\mathbb{N}}$, denoted (Γ_1, Γ_2)-Det_W^X, is given by the axiom scheme $\mathsf{Det}_W^X(A,B)$ where $A(f)$ is in Γ_1 and $B(g)$ is in Γ_2 (if $\Gamma_1 = \Gamma_2 = \Gamma$, we will simply write Γ-Det_W^X.)

2. Likewise (Δ^0_n, Δ^0_m)-Det^X_W, (Γ, Δ^0_n)-Det^X_W and (Δ^0_n, Γ)-Det^X_W are defined as in Definition 2.1.

Also, we will omit the superscript $\mathbb{N}$ for determinacy schemes in the *Baire space* and replace the superscript $\{0,1\}$ with $*$ for determinacy schemes in the *Cantor space*.

Remark 2.3. It is easily verified that Γ-Det^X and $(\neg\Gamma)$-Det^X are equivalent principles over RCA_0. A similar result holds for Lipschitz and Wadge determinacy: (Γ_1, Γ_2)-$\mathsf{Det}^X_{L/W}$ and $(\neg\Gamma_1, \neg\Gamma_2)$-$\mathsf{Det}^X_{L/W}$ are equivalent over RCA_0 and in this case the proof is trivial, because $G^X_{L/W}(A, B)$ and $G^X_{L/W}(\neg A, \neg B)$ are essentially the same game.

Remark 2.4. Lipschitz and Wadge games can be naturally reduced to *classical* infinite games. This reduction is effective and checkable in RCA_0. However, there is a price to pay: a possible increase of the pay-off set complexity. Fix $X \subseteq \mathbb{N}$ nonempty and consider formulas $A(f), B(g)$ with $f, g \in X^{\mathbb{N}}$. Let us start by analysing the Lipschitz case. The resulting plays in a Lipschitz game $f = h^{\mathrm{I}}, g = h^{\mathrm{II}}$ can be defined by composition from the resulting play in a classical infinite game h. In fact, $h^{\mathrm{I}}(n) = h(2n)$ and $h^{\mathrm{II}}(n) = h(2n+1)$. Put

$$\mathsf{Trans}_L(A, B)(h) \equiv \neg(A(h^{\mathrm{I}}) \leftrightarrow B(h^{\mathrm{II}})),$$

where h ranges over functions in $X^{\mathbb{N}}$. It is easy to check that $\mathsf{Det}^X(\mathsf{Trans}_L(A, B)) \to \mathsf{Det}^X_L(A, B)$.

Since $\neg(A(h^{\mathrm{I}}) \leftrightarrow B(h^{\mathrm{II}}))$ is equivalent to $(A(h^{\mathrm{I}}) \vee B(h^{\mathrm{II}})) \wedge (\neg A(h^{\mathrm{I}}) \vee \neg B(h^{\mathrm{II}}))$, this reduction allows one to infer Lipschitz Γ-determinacy from determinacy for $\Gamma \wedge \neg\Gamma$ sets (We are assuming that Γ is closed under conjunction and disjunctions.) This provides us with a first upper bound on the strength of Lipschitz Γ-determinacy. This upper bound needn't be, however, sharp. Consider, for example, $\Gamma = \Sigma^0_4$. By a result of Louveau and Saint-Raymond [6] second order arithmetic proves Σ^0_4-Det_L, whereas by a result of Martin [7] second order arithmetic does not prove $\Sigma^0_4 \wedge \Pi^0_4$-$\mathsf{Det}$.

It should be noted that in [9] Nemoto studied the logical strength of several axiom schemes formalizing the determinacy of classical infinite games whose pay-off sets are of the form $\mathrm{Sep}(\Gamma_1, \Gamma_2) = \{(\mathrm{A} \wedge \neg\mathrm{B}) \vee (\neg\mathrm{A} \wedge \mathrm{C}) : \mathrm{A} \in \Gamma_1, \mathrm{B}, \mathrm{C} \in \Gamma_2\}$. Since $\neg(A(h^{\mathrm{I}}) \leftrightarrow B(h^{\mathrm{II}}))$ is equivalent to $(A(h^{\mathrm{I}}) \wedge \neg B(h^{\mathrm{II}})) \vee (\neg A(h^{\mathrm{I}}) \wedge B(h^{\mathrm{II}}))$, a Lipschitz game whose pay-off sets have complexity (Γ_1, Γ_2) unravels to an infinite game of complexity $\mathrm{Sep}(\Gamma_1, \Gamma_2)$. Thus, using the results in [9], one can obtain upper bounds on the strength of Lipschitz (Γ_1, Γ_2)-determinacy. Again, these upper bounds needn't be sharp, for the translations of (Γ_1, Γ_2) Lipschitz games give rise to special cases of $\mathrm{Sep}(\Gamma_1, \Gamma_2)$ games.

As to the Wadge case, we have to modify the way we recover the resulting plays because now player II is allowed to pass. To this end, given $h \in (X^+)^{\mathbb{N}}$ put $h^{\mathrm{I},W}(n) = h(2n) \dot{-} 1$ and $h^{\mathrm{II},W}(n) = h(2 \cdot \mathit{move}(h)(n) + 1)) - 1$, where $\mathit{move}(h)$ is given by the recursive equations

$$\begin{aligned} \mathit{move}(h)(0) &= \mu i\, [h(2i+1) \neq 0], \\ \mathit{move}(h)(n+1) &= \mu i\, [i > \mathit{move}(h)(n) \wedge h(2i+1) \neq 0]. \end{aligned}$$

Let $\mathit{Inf}(h)$ denote the Π^0_2 formula $\forall n\, \exists k\, (k > n \wedge h(2k+1) \neq 0)$ expressing that player II plays infinitely often and put

$$\mathsf{Trans}_W(A, B)(h) \equiv \mathit{Inf}(h) \to \neg(A(h^{\mathrm{I},W})) \leftrightarrow B(h^{\mathrm{II},W})).$$

It is easy to see that $\mathsf{Det}^{X^+}(\mathsf{Trans}_W(A, B)) \to \mathsf{Det}^X_W(A, B)$.

Roughly speaking, the previous reduction allows one to infer Wadge Γ-determinacy from determinacy for $\mathit{max}(\Sigma^0_2, \Gamma \wedge \neg\Gamma)$ sets. The possible increase of the pay-off set complexity is now much worse due to the presence of the Π^0_2 formula $\mathit{Inf}(h)$ in the translation of the game. For example, the translation of a clopen Wadge game would already give rise to a Σ^0_2 classical game.

3. Determinacy for clopen sets

In [17] Wadge developed a careful level-by-level analysis of the structure of Wadge degrees (i.e. equivalence classes relative to $\leq_W$) in the Baire space below $\mathbf{\Delta}^0_2$. In this and the following section, we will show that this topological analysis can be adapted to prove Lipschitz and Wadge determinacy for the first levels of the Hausdorff difference hierarchy in the Cantor space. In the present section we will deal with clopen determinacy. Our starting point is the standard representation of closed sets of the Cantor space as sets of paths of binary trees.

A set $T \subseteq X^{<\mathbb{N}}$ is called a *tree* over X if T is closed under initial segments, i.e. $s \in T$ and $t \subseteq s$ imply $t \in T$. We call the elements of T the *nodes* of T. A tree is *infinite* if, for any n, there exists $s \in T$ with $|s| = n$, i.e. if the set of nodes of T is infinite. If $S \subseteq X^{<\mathbb{N}}$ is a tree over X and $S \subseteq T$, then S is called a subtree of T.

Fix any tree $T \subseteq X^{<\mathbb{N}}$. A node $s \in T$ is called *terminal* if it has no proper extension in T, i.e. if $\forall a \in X$ $(s * \langle a \rangle \notin T)$. A function $f \in X^{\mathbb{N}}$ is called a *path* of T if $\forall n \in \mathbb{N}$ $(f[n] \in T)$. The *body* of T is written as $[T]$ and is the set of all paths of T, i.e.

$$[T] = \{f \in X^{\mathbb{N}} : \forall n \in \mathbb{N}\ f[n] \in T\}.$$

Proposition 3.1 ([13], Lemma VI.1.5)**.** *For each formula $\varphi(f) \in \Pi^0_1$, RCA_0 proves that there is a tree $T \subseteq X^{<\mathbb{N}}$ satisfying that $[T] = \{f \in X^{\mathbb{N}} : \varphi(f)\}$.*

By abuse of language, we will use set theoretic notations to mean the arithmetic formula expressing the corresponding set. For instance, an expression of the form $f \in [T]$ denotes the Π^0_1 formula $\forall n\,(f[n] \in T)$ expressing that f is a path of T and, accordingly, an expression of the form $f \in [T] - [S]$ is to be understood as the $\Pi^0_1 \wedge \Sigma^0_1$ formula expressing that f is a path of T and is not a path of S.

Definition 3.2. Fix $X \subseteq \mathbb{N}$ nonempty and consider a tree $T \subseteq X^{<\mathbb{N}}$. We say that T *defines a clopen set* if there exists another tree $T' \subseteq X^{<\mathbb{N}}$ such that $\forall f \in X^{\mathbb{N}}\,(f \notin [T] \leftrightarrow f \in [T'])$. We say that T *properly defines a finitely decidable set* if $\exists k\, \forall f \in X^{\mathbb{N}}\,(f \in [T] \leftrightarrow f[k] \in T)$.

It is easily proved in RCA_0 that if T is a binary tree that properly defines a finitely decidable set then T defines a clopen set. The converse can be proved in WKL_0 and, so, over WKL_0 both notions are equivalent. Even more, we have

Lemma 3.3. *Over RCA_0, WKL_0 is equivalent to the assertion "every binary tree defining a clopen set properly defines a finitely decidable set."*

Proof. First, let us reason in WKL_0 and consider binary trees T and T' such that $\forall f \in X^{\mathbb{N}}\,(f \notin [T] \leftrightarrow f \in [T'])$. Clearly, $T \cap T'$ is a binary tree with no path. By WKL_0, $T \cap T'$ must be finite. Pick $k \in \mathbb{N}$ such that all sequences in $T \cap T'$ have length at most k. Then, $\forall f \in X^{\mathbb{N}}\,(f \in [T] \leftrightarrow f[k+1] \in T)$. Second, reason in RCA_0 and assume WKL_0 fails. Let T_1 be an infinite binary tree with no path. Then, it is easy to see that T_1 defines a clopen set but T_1 does not properly define a finitely decidable set. □

Definition 3.4. A tree $T \subseteq X^{<\mathbb{N}}$ is said to be *pruned* if every sequence of T lies on a path of T, i.e. $\forall s \in X^{<\mathbb{N}}\,(s \in T \rightarrow \exists f \in X^{\mathbb{N}}\,(s \subset f \wedge f \in [T]))$.

It is well known that the assertion that every tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ can be pruned (that is to say, for every tree there exists some pruned subtree with the same set of paths) is equivalent over RCA_0 to Π^1_1-CA_0 (see, e.g., Lemma VI.4.4 of [13]). Here we show that if we restrict ourselves to binary trees then we can prune a tree at a lower price. We first need the following lemma asserting that Π^0_1 formulas are closed in WKL_0 under existential quantifiers of the form $\exists f \in 2^{\mathbb{N}}$.

Lemma 3.5 (see [13], Lemma VIII.2.4, or [11], Lemma 3.2)**.** *Let ψ be a Π^0_1 formula. Within WKL_0, $\exists f \in 2^{\mathbb{N}}\, \psi(f)$ is equivalent to a Π^0_1 formula.*

Lemma 3.6.

1. *Let $\varphi(f) \in \Pi^0_1$. ACA_0 proves that there is a pruned binary tree T such that $[T] = \{f \in 2^{\mathbb{N}} : \varphi(f)\}$.*
2. *Let $\varphi(f) \in \Sigma^0_1$ and $\psi(f) \in \Pi^0_1$. WKL_0 proves that if $\forall f \in 2^{\mathbb{N}}(\varphi(f) \leftrightarrow \psi(f))$ then there exists a pruned binary tree T such that $[T] = \{f \in 2^{\mathbb{N}} : \varphi(f)\}$.*

Proof.
(1) We work in ACA_0. By Proposition 3.1, let S be a binary tree such that $[S] = \{f \in 2^{\mathbb{N}} : \varphi(f)\}$. It suffices to consider the tree $T = \{s \in 2^{<\mathbb{N}} : s \in S \wedge \exists f \in 2^{\mathbb{N}}\, (f \in [S] \wedge s \subset f)\}$. The existence of T follows by Π^0_1-comprehension (which is available in ACA_0), for the formula defining T is equivalent to a Π^0_1 formula by Lemma 3.5. It is clear that T is a pruned subtree of S with $[S] = [T]$.

(2) We work in WKL_0. Let S and S' be binary trees such that $[S] = \{f \in 2^{\mathbb{N}} : \psi(f)\}$ and $[S'] = \{f \in 2^{\mathbb{N}} : \neg\varphi(f)\}$. Let us consider the formulas

$$A(s) \equiv\ s \in S \wedge \exists f \in 2^{\mathbb{N}}\, (f \in [S] \wedge s \subset f) \quad \text{and} \quad B(s) \equiv\ s \in S \wedge \exists t\, (s \subseteq t \wedge t \notin S').$$

It is clear that $B(s) \in \Sigma^0_1$ and it follows from Lemma 3.5 that $A(s)$ is equivalent to a Π^0_1 formula. In addition, we have
Claim 3.6.1. $\forall s \in 2^{<\mathbb{N}}\, (A(s) \leftrightarrow B(s))$.

Proof. Pick $s_1 \in 2^{<\mathbb{N}}$ satisfying $A(s_1)$. There exists $g \in [S]$ such that $s_1 \subset g$. Then, $g \notin [S']$ since $[S] \cap [S'] = \emptyset$. Consequently, there must exist some $k > |s_1|$ such that $g[k] \notin S'$. We have $s_1 \subseteq g[k]$ and $g[k] \notin S'$ and so $B(s_1)$ holds. Now pick $s_2 \in 2^{<\mathbb{N}}$ satisfying $B(s_2)$. There exists t_1 such that $s_2 \subseteq t_1$ and $t_1 \notin S'$. Define $h : \mathbb{N} \to \{0, 1\}$ by putting $h(i) = t_1(i)$ if $i < |t_1|$ and $h(i) = 0$ otherwise. Clearly, $h \notin [S']$ and so h must be in $[S]$. But then we have $s_2 \subset h$ and $h \in [S]$ and hence $A(s_2)$ holds. □

It follows from the claim that $\{s : A(s)\}$ exists by Δ^0_1 comprehension. It is immediate to check that $A(s)$ defines a pruned binary tree, T, with $[T] = [S]$, as required. □

Remark 3.7. WKL_0 proves that if $T \subseteq 2^{<\mathbb{N}}$ is a pruned tree defining a clopen set then there is a unique pruned binary tree T' such that $[T] \cap [T'] = \emptyset$ and $[T] \cup [T'] = 2^{\mathbb{N}}$. Now, assume that the set $T \cap T'$ is nonempty (which amounts to assuming that $[T]$ is a nontrivial clopen, that is $[T] \neq \emptyset$ and $[T] \neq 2^{\mathbb{N}}$). Note that $T \cap T'$ is a binary tree with no path, as T and T' do not have any common paths. Hence, $T \cap T'$ must be finite by WKL_0. By Δ^0_0 induction, we obtain that there exists $h_T = \max\{|s| :\ s \in T \cap T'\}$, the height of $T \cap T'$. By the uniqueness of T', h_T is well defined and we call it the *stem* of T. The following lemma sums up the main properties of the stem h_T (its proof easily follows from the very definition of h_T.)

Lemma 3.8. (RCA_0) *Let $T, T' \subseteq 2^{<\mathbb{N}}$ be pruned trees such that $T \cap T'$ is finite and non empty, $[T] \cap [T'] = \emptyset$ and $[T] \cup [T'] = 2^{\mathbb{N}}$. Then $h_T = \max\{|s| : s \in T \cap T'\}$ satisfies the following properties:*

1. $\forall f \in 2^{\mathbb{N}} \, (f \in [T] \leftrightarrow f[h_T + 1] \in T)$.
2. *If $s \in T \cap T'$ satisfies $|s| = h_T$ then one of the following holds:*

$$\forall f \in 2^{\mathbb{N}} \, [(s * \langle 0 \rangle \subset f \to f \in [T]) \wedge (s * \langle 1 \rangle \subset f \to f \notin [T])], \textit{or}$$

$$\forall f \in 2^{\mathbb{N}} \, [(s * \langle 0 \rangle \subset f \to f \notin [T]) \wedge (s * \langle 1 \rangle \subset f \to f \in [T])].$$

Proof. Part (1) follows easily from the very definition of h_T, so we only prove part (2). Given $s \in T \cap T'$ such that $|s| = h_T$, we consider two cases:
Case 1: $s * \langle 0 \rangle \in T$.
Since $|s * \langle 0 \rangle| = h_T + 1$ and $s * \langle 0 \rangle \in T$, it follows from part (1) that $s * \langle 0 \rangle \subset f \to f \in [T]$, for each $f \in 2^{\mathbb{N}}$. Now pick $f \in 2^{\mathbb{N}}$ satisfying $s * \langle 1 \rangle \subset f$. We must show that $f \notin [T]$. Notice that $s * \langle 0 \rangle \notin T'$, for otherwise $s * \langle 0 \rangle$ would be in $T \cap T'$, contradicting the maximality of h_T. Hence, since $s \in T'$ and T' is a pruned tree, $s * \langle 1 \rangle$ must be in T'. But then $s * \langle 1 \rangle$ is not in T, for otherwise $s * \langle 1 \rangle$ would be in $T \cap T'$, again contradicting the maximality of h_T. Therefore, $f \notin [T]$ as required.
Case 2: $s * \langle 0 \rangle \notin T$.
Since $s \in T$ and T is a pruned tree, $s * \langle 1 \rangle$ must be in T. By reasoning as in the previous case, we obtain that for each $f \in 2^{\mathbb{N}}$, $s * \langle 1 \rangle \subset f \to f \in [T]$ and $s * \langle 0 \rangle \subset f \to f \notin [T]$. □

We are now in a position to prove determinacy for clopen games. For notational convenience, from now on we will simply write $G_{L/W}(A, B)$ instead of $G^{\{0,1\}}_{L/W}(A, B)$.

Proposition 3.9.

1. WKL_0 *proves* Δ^0_1-Det^*_L.
2. RCA_0 *proves* (Δ^0_1, Σ^0_n)-Det^*_W, *for every* $n > 0$.

Proof. (1) We work in an arbitrary model of WKL_0. Consider $A(f), B(g) \in \Sigma^0_1$ and $A'(f), B'(g) \in \Pi^0_1$ satisfying that $\forall f \in 2^{\mathbb{N}} \, (A(f) \leftrightarrow A'(f))$ and $\forall g \in 2^{\mathbb{N}} \, (B(g) \leftrightarrow B'(g))$. Note that if some of A, $\neg A$, B or $\neg B$ defines the empty set then the game $G_L(A, B)$ is determined. In fact, if $\mathsf{Empty}(\varphi)$ denotes the formula $\neg \exists f \in 2^{\mathbb{N}} \varphi(f)$ and $\mathsf{Tot}(\varphi)$ denotes the formula $\forall f \in 2^{\mathbb{N}} \varphi(f)$, the following facts are readily checkable in RCA_0:

$$(\mathsf{Empty}(A) \wedge \neg \mathsf{Tot}(B)) \vee (\mathsf{Tot}(A) \wedge \neg \mathsf{Empty}(B)) \to \exists \sigma_{\mathrm{II}} \, \forall \sigma_{\mathrm{I}} \, (A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}}))$$
and
$$(\neg \mathsf{Tot}(A) \wedge \mathsf{Tot}(B)) \vee (\neg \mathsf{Empty}(A) \wedge \mathsf{Empty}(B)) \to \exists \sigma_{\mathrm{I}} \, \forall \sigma_{\mathrm{II}} \, \neg (A(\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}}))$$

Hence, we may assume that none of A, $\neg A$, B or $\neg B$ defines the empty set and then by Lemma 3.6, there are nonempty pruned binary trees $S, S', T, T' \subseteq 2^{<\mathbb{N}}$ such that

$$[S] = \{f \in 2^{\mathbb{N}} : A(f)\} \text{ and } [S'] = \{f \in 2^{\mathbb{N}} : \neg A(f)\}$$

$$[T] = \{g \in 2^{\mathbb{N}} : B(g)\} \text{ and } [T'] = \{g \in 2^{\mathbb{N}} : \neg B(g)\}$$

In order to show that $G_L([S],[T])$ is determined, it suffices to compare the stems h_T and h_S. Indeed, if $h_S > h_T$ then we can define a winning strategy for player I; whereas if $h_S \leq h_T$ then player II will win the game.

Firstly, suppose $h_S > h_T$. Pick $s_1 \in S \cap S'$ such that $|s_1| = h_S$ and assume that

$$\forall f \in 2^{\mathbb{N}} \left[(s_1 * \langle 0 \rangle \subset f \to A(f)) \wedge (s_1 * \langle 1 \rangle \subset f \to \neg A(f)) \right] \tag{$\dagger$}$$

(If the other possibility in Lemma 3.8 holds, the definition is to be modified accordingly.) A strategy for player I, σ_{I}, can be described informally as follows: I plays using s_1 during his first h_S moves and then he decides his next move depending on player II's previous h_S moves. If II has played outside T then player I will play 0 to stay inside S; if II has played inside T then she will remain inside T forever ($h_T < h_S$) and I will pick 1 to go outside S. Formally, put

$$\sigma_{\mathrm{I}}(\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle) = \begin{cases} s_1(n) & \text{if } n < h_S \\ 1 & \text{if } n = h_S \text{ and } \langle y_0, \ldots, y_{n-1} \rangle \in T \\ 0 & \text{if } n = h_S \text{ and } \langle y_0, \ldots, y_{n-1} \rangle \notin T \\ 0 & \text{if } n > h_S \end{cases}$$

It is clear that σ_{I} exists by Δ^0_1-comprehension and it follows from Lemma 3.8 and the assumption ($\dagger$) that σ_{I} is winning for player I.

Secondly, suppose $h_S \leq h_T$. Let $s_2 \in T \cap T'$ be such that $|s_2| = h_T$ and assume that

$$\forall g \in 2^{\mathbb{N}} \left[(s_2 * \langle 0 \rangle \subset g \to B(g)) \wedge (s_2 * \langle 1 \rangle \subset g \to \neg B(g)) \right] \tag{$\dagger\dagger$}$$

(If the other possibility in Lemma 3.8 holds, the definition should be modified accordingly.) We can construct a winning strategy σ_{II} for player II as follows: II plays using s_2 during her first h_T moves and then she decides her next move depending on player I's previous $h_T + 1$ moves. If I has played inside S then he will remain inside S forever ($h_S \leq h_T$) and player II will play 0 to remain inside T; otherwise II will pick 1 to go outside T.
Clearly, σ_{II} exists by Δ^0_1-comprehension and σ_{II} is winning for player II.

(2) We work in an arbitrary model of RCA_0. Consider $A(f) \in \Sigma^0_1$, $A'(f) \in \Pi^0_1$ and $B(f) \in \Sigma^0_n$ satisfying that $\forall f \in 2^{\mathbb{N}}(A(f) \leftrightarrow A'(f))$. Reasoning as in part (1), we can deal with trivial cases where $\mathsf{Empty}(B)$ or $\mathsf{Tot}(B)$ holds. So, we may assume that there are $g_{in}, g_{out} \in 2^{\mathbb{N}}$ satisfying $B(g_{in})$ and $\neg B(g_{out})$, respectively. By Proposition 3.1, there are binary trees $S, S' \subseteq 2^{<\mathbb{N}}$ such that

$$[S] = \{f \in 2^{\mathbb{N}} : A(f)\} \text{ and } [S'] = \{f \in 2^{\mathbb{N}} : \neg A(f)\} .$$

We must show that the Wadge game $G_W([S], B)$ is determined. Observe that the binary tree $S \cap S'$ cannot have any path. As a consequence, we can define a winning strategy for player II. The idea is simple: while player I plays inside $S \cap S'$ player II passes; and when player I leaves $S \cap S'$ (this has to happen sooner or later as $S \cap S'$ has no path) player II plays accordingly by using either g_{in} or g_{out}. In order to give the formal definition of the strategy, recall that for player II we identify passing with picking the number 0, and for both players we identify playing i with picking $i+1$. Having this in mind, given any sequence of odd length $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n \rangle$, we put

$$\sigma_{\mathrm{II}}(s) = \begin{cases} 0 & \text{if } \langle x_0 \dot{-} 1, \ldots, x_n \dot{-} 1 \rangle \in S \cap S' \\ g_{out}(n-k)+1 & \text{if } \langle x_0 \dot{-} 1, \ldots, x_n \dot{-} 1 \rangle \notin S \text{ and } k = \mu j \ (\langle x_0 \dot{-} 1, \ldots, x_j \dot{-} 1 \rangle \notin S) \\ g_{in}(n-k)+1 & \text{if } \langle x_0 \dot{-} 1, \ldots, x_n \dot{-} 1 \rangle \notin S' \text{ and } k = \mu j \ (\langle x_0 \dot{-} 1, \ldots, x_j \dot{-} 1 \rangle \notin S') \end{cases}$$

Clearly, σ_{II} exists by Δ^0_1 comprehension and it is easy to see that σ_{II} is winning for player II. $\square$

Remark 3.10. The schema of Σ^0_1 induction has not been used so far. Thus, it follows from the proof of Proposition 3.9 that WKL^*_0 already proves Δ^0_1-Det^*_L. (The system WKL^*_0 is essentially WKL_0 with Σ^0_1 induction weakened to Δ^0_0 induction, see Definition X.4.1 of [13] for a definition.) By a result of Nemoto (see Proposition 3.1 of [9]), WKL^*_0 proves Δ^0_1-Det^*. By Remark 2.4, a clopen Lipschitz game can be effectively reduced to a clopen infinite game. Putting these two facts together, we obtain another proof of part 1 of Proposition 3.9. However, the topological character of our proof of Proposition 3.9 makes it, we think, quite simple and it implicitly establishes the following fact:

Claim 3.10.1. (RCA_0) Let S and T be pruned binary trees defining two clopen sets and such that, under the conditions of Lemma 3.8, their stems h_S and h_T respectively are well defined. Then Player I wins $G_L([S],[T])$ iff $h_S > h_T$, and player II wins $G_L([S],[T])$ iff $h_T \geq h_S$. □

We are now ready for our first reversal. Characterizing the Weak Köning Lemma by means of Lipschitz/Wadge determinacy was left open in [5]. Actually, Problem 4.17 in [5] asks

- *Is* WKL_0 *equivalent over* RCA_0 *to* Δ^0_1-Det^*_L*?*

Recently, Chan [4] has proven WKL_0 to be equivalent over RCA_0 to $(\Sigma^0_1\wedge\Pi^0_1, \Delta^0_1)$-$\mathsf{Det}^*_L$, thus giving a first partial answer to Problem 4.17 of [5]. Here we will give a complete answer to that problem by showing that Δ^0_1-Det^*_L suffices to recover WKL_0 over RCA_0.

Theorem 3.11. *The following are equivalent over* RCA_0*:*

1. WKL_0.
2. Δ^0_1-Det^*_L.

Proof. We only prove that (2) implies (1). We work in an arbitrary model of $\mathsf{RCA}_0 + \Delta^0_1$-$\mathsf{Det}^*_L$. Let h_1, $h_2 \in \mathbb{N}$ be such that $h_1 > h_2 + 1$ and $h_2 > 0$. For each $k \in \mathbb{N}$, let $0^{(k)}$ denote the finite sequence $s \in 2^{<\mathbb{N}}$ such that $|s| = k$ and for all $j < k$, $s(j) = 0$. For $i = 1, 2$, we denote by N_i the set

$$\{s \in 2^{<\mathbb{N}} :\ 0^{(h_i)} \subseteq s \vee s \subseteq 0^{(h_i)}\}$$

(Note that N_1 and N_2 are pruned trees with stems $h_{N_1} = h_1$ and $h_{N_2} = h_2$, and by Claim 3.10.1 player I has a winning strategy in the game $G_L([N_1],[N_2])$.) Towards a contradiction, let us assume that WKL_0 fails and consider an infinite binary tree T_0 without paths. For each $i = 1, 2$ we define a binary tree T_i as follows:

$$s \in T_i \Leftrightarrow \left\{ \begin{array}{l} s \in T_0 \vee \\ \exists t_0\, \exists t\, (s = t_0 * t \wedge t_0 \in T_0 \wedge t_0 * \langle 0\rangle \notin T_0 \wedge t_0 * \langle 1\rangle \notin T_0 \wedge t \in N_i) \end{array} \right.$$

i.e., T_i extends each terminal node of T_0 with a copy of N_i. For $i = 1, 2$, T_i is Δ^0_1-definable and therefore it does exist by Δ^0_1-comprehension. Let us see that T_i defines a clopen set (i.e. $[T_i]$ is Δ^0_1-definable). For each $f \in 2^{\mathbb{N}}$,

$$\begin{aligned} f \in [T_i] &\Leftrightarrow \forall k\, (f[k] \in T_i) \\ &\Leftrightarrow \exists m \exists k < m \exists s \left\{ \begin{array}{l} f[k] \in T_0 \wedge f[k] * \langle 0\rangle \notin T_0 \wedge f[k] * \langle 1\rangle \notin T_0 \wedge \\ f[m] = f[k] * s \wedge s \in N_i \wedge |s| > h_i \end{array} \right. \end{aligned}$$

By Δ^0_1-Det^*_L the game $G_L([T_1],[T_2])$ is determined. So either player I or player II must have a winning strategy.

Let us assume that player I has a winning strategy $\sigma : \mathrm{Seq}_{\mathrm{even}} \to \{0,1\}$. Consider the strategy for player II, $\tau_0 : \mathrm{Seq}_{\mathrm{odd}} \to \{0,1\}$, that simply mimics player I's moves. That is, for each $s \in \mathrm{Seq}_{\mathrm{odd}}$, $\tau_0(s) = s(|s|-1)$. Let $f = \sigma \otimes^{\mathrm{I}} \tau_0$ be the corresponding player I's resulting play and let $\mathsf{InfMany}(t, T_0)$ be the following Π^0_1 formula expressing "t has infinitely many extensions in T_0:"

$$\forall u\,(|t| \leq u \to \exists s\,(s \in T_0 \wedge t \subseteq s \wedge |s| = u)).$$

Claim 3.11.1. $\forall k\, \mathsf{InfMany}(f[k], T_0)$.

Proof. We show that $\forall k\, \mathsf{InfMany}(f[k], T_0)$ by Π^0_1-induction on k.
$\underline{k=0}$: By hypothesis T_0 is infinite, so $\mathsf{InfMany}(f[0], T_0)$ holds.

$\underline{k \to k+1}$: By way of contradiction, assume that $\neg\mathsf{InfMany}(f[k+1], T_0)$ holds. Then there is $k_0 > k+1$ such that $\forall s\,(s \in T_0 \wedge f[k+1] \subseteq s \to |s| \leq k_0)$. Since by induction hypothesis we have $\mathsf{InfMany}(f[k], T_0)$, there must be some t such that

$$(\dagger) \quad t \in T_0 \wedge f[k] \subseteq t \wedge |t| > k_0 + h_1 \wedge t * \langle 0 \rangle \notin T_0 \wedge t * \langle 1 \rangle \notin T_0.$$

In order to verify ($\dagger$), take $s \in T_0$ such that $f[k] \subseteq s \wedge |s| > k_0 + h_1$. We will show that s can be extended up to a terminal node of T_0. To this end, define $g : \mathbb{N} \to \mathbb{N}$ by primitive recursion as follows:

$$\begin{array}{ll} g(j) = s(j) & \text{if } j < |s|, \text{ and} \\ g(j+1) = i & \text{if } j \geq |s|, \text{ where } i = 0 \text{ if } g[j] * \langle 0 \rangle \in T_0 \text{ and } i = 1 \text{ otherwise.} \end{array}$$

Then there exists m such that $g[m] \notin T_0$ (otherwise g would be a path of T_0). Let j_0 be the least m such that $g[m] \notin T_0$. Taking $t = g[j_0 - 1]$, we get that $t * \langle 0 \rangle \notin T_0 \wedge t * \langle 1 \rangle \notin T_0$, but $t \in T_0 \wedge |t| > |s| > k_0 + h_1$ and thus ($\dagger$) follows.
Pick $t \in T_0$ as in ($\dagger$). We define a strategy τ_t for player II such that

$$\tau_t(s) = \begin{cases} t(j) & \text{if } |s| = 2j+1 \text{ and } j < |t|; \\ 0 & \text{if } |s| > 2|t|,\ s = s_1 \otimes s_2 \text{ and } s_1 \in T_1; \\ 1 & \text{otherwise} \end{cases}$$

where $s_1 \otimes s_2$ denotes the finite sequence $\langle s_1(0), s_2(0), s_1(1), s_2(1), \ldots \rangle$. (Informally, player II plays using t until she reaches the corresponding terminal node and then she will remain inside T_2 if player I has played inside T_1 or she will leave T_2 otherwise.) On the one hand, if $\sigma \otimes^{\mathrm{I}} \tau_t \in [T_1]$ then, obviously, $\sigma \otimes^{\mathrm{II}} \tau_t = t * g \in [T_2]$ (where, $\forall i\,(g(i) = 0)$). On the other hand, let us observe that, by definition, $(\sigma \otimes^{\mathrm{I}} \tau_t)[k] = f[k] \subseteq t$ and, therefore,

$$(\sigma \otimes^{\mathrm{I}} \tau_t)(k) = \sigma(f[k] \otimes t[k]) = \sigma(f[k] \otimes f[k]) = f(k)$$

(recall that τ_0 just mimics σ). Thus, $(\sigma \otimes^{\mathrm{I}} \tau_t)[k+1] = f[k+1]$. As a consequence, if $\sigma \otimes^{\mathrm{I}} \tau_t \notin [T_1]$ then, there exists $j < k_0 + h_1 < |t|$ such that $(\sigma \otimes^{\mathrm{I}} \tau_t)[j] \notin T_1$ and $(\sigma \otimes^{\mathrm{II}} \tau_t)(|t|) = 1$, which gives us $(\sigma \otimes^{\mathrm{II}} \tau_t)[|t|] = t * \langle 1 \rangle \notin T_2$. We have shown this way that $\sigma \otimes^{\mathrm{I}} \tau_t \in [T_1]$ iff $\sigma \otimes^{\mathrm{II}} \tau_t \in [T_2]$, but we are assuming that σ is a winning strategy for player I and we get a contradiction. $\square$

It follows from previous Claim that f is a path of T_0, which contradicts our assumption on T_0. Therefore, player I cannot have a winning strategy in $G_L([T_1], [T_2])$.

Now suppose that player II has a winning strategy, say τ. For each $s \in 2^{<\mathbb{N}}$ with $|s| \geq 1$, let us denote by $s \otimes^{\mathrm{II}} \tau$ the unique finite sequence t such that $|t| = |s|$ and, for all $j < |t|-1$, $t(0) = \tau(\langle s(0)\rangle)$ and $t(j+1) = \tau(\langle s(0), t(0), \ldots, s(j), t(j), s(j+1)\rangle)$.

Let $\mathsf{Cyclic}(s)$ be the formula:

$$s \in 2^{<\mathbb{N}} \wedge |s| \geq 1 \wedge \exists w \, \left[|w| > 1 \wedge (w)_0 = s \wedge (w)_{|w|-1} = s \wedge \forall i < |w|-1 \, ((w)_{i+1} = (w)_i \otimes^{\mathrm{II}} \tau)\right]$$

where $(w)_i$ denotes $w(i)$, i.e. the $(i+1)$-th element of the finite sequence w. Observe that $\mathsf{Cyclic}(s)$ is a Δ^0_1-formula. Indeed, if $\mathsf{Cyclic}(s)$ holds then the length of the finite sequence w in the above formula can be bounded by $2^{|s|}+1$, and it is routine to check that there are Δ^0_1-definable functions $\mathsf{Bd}_1(x) = y$ and $\mathsf{Bd}_2(x, z) = y$ such that provably in RCA_0, we have $\forall s, k \, (s \in 2^{<\mathbb{N}} \wedge |s| = k \to s \leq \mathsf{Bd}_1(k))$ and $\forall w, k, a \, (w \in \mathbb{N}^{<\mathbb{N}} \wedge |w| = k \wedge \forall i < k \, (w(i) \leq a) \to w \leq \mathsf{Bd}_2(k, a))$.

Claim 3.11.2.

($\flat$) If $s \in T_0$ and $|s| \geq 1$ then $s \otimes^{\mathrm{II}} \tau \in T_0$.

($\sharp$) Let $S_0 = \{s : s \in T_0 \wedge (s = \langle\rangle \vee \mathsf{Cyclic}(s))\}$. Then S_0 is a tree and $\forall s \, (s \in S_0 \to s * \langle 0\rangle \in S_0 \vee s * \langle 1\rangle \in S_0)$.

Proof. We prove ($\flat$) by contradiction. Suppose that there is $s_1 \in T_0$ with $|s_1| \geq 1$ such that $s_1 \otimes^{\mathrm{II}} \tau \notin T_0$. Reasoning as in the proof of Claim 3.11.1, we can extend s_1 to a terminal node of T_0, i.e., there exists t such that

$$t \in T_0 \wedge s_1 \subseteq t \wedge t * \langle 0\rangle \notin T_0 \wedge t * \langle 1\rangle \notin T_0.$$

We define a strategy σ_t for player I by putting (for every $s', s'' \in 2^{<\mathbb{N}}$ such that $|s'| = |s''|$):

$$\sigma_t(s' \otimes s'') = \begin{cases} t(|s'|) & \text{if } |s'| < |t|; \\ 0 & \text{if } |t| \leq |s'| < |t| + h_1-1; \\ 1 & \text{if } |s'| = |t| + h_1-1 \wedge s'' \in T_2; \\ 0 & \text{otherwise.} \end{cases}$$

(In words, player I plays according to t until he reaches a terminal node of T_0, after that he plays 0 h_1-1 times and then he decides whether or not to remain inside T_1: if player II has played inside T_2 after her first h_1-1 moves then she will remain inside T_2 forever ($h_2+1 < h_1$) and player I picks 1 to play outside T_1; otherwise player I picks 0.) Then $\sigma_t \otimes^{\mathrm{I}} \tau \in [T_1]$ iff $\sigma_t \otimes^{\mathrm{II}} \tau \notin [T_2]$ and so τ would not be a winning strategy for player II.

Now let us prove ($\sharp$). It is straightforward to check that S_0 is a tree. Pick $s \in S_0$ with $|s| \geq 1$ (we omit the case $s = \langle\rangle$ which is simpler). Firstly, we will show that either $s * \langle 0\rangle \in T_0$ or $s * \langle 1\rangle \in T_0$. Towards a contradiction, assume not. Since $\mathsf{Cyclic}(s)$ holds, there exists w such that

$$|w| > 1 \wedge (w)_0 = s \wedge (w)_{|w|-1} = s \wedge \forall i < |w|-1 \, ((w)_{i+1} = (w)_i \otimes^{\mathrm{II}} \tau).$$

Using ($\flat$), it follows by Δ^0_0-induction that $\forall i < |w| \, ((w)_i \in T_0)$. As a consequence, taking $s_1 = (w)_{|w|-2}$ we get $s_1 \in T_0$ such that $s = s_1 \otimes^{\mathrm{II}} \tau$. Now we reason as in the proof of ($\flat$). Since $s_1 \in T_0$, there exists t such that

$$t \in T_0 \wedge s_1 \subseteq t \wedge t * \langle 0\rangle \notin T_0 \wedge t * \langle 1\rangle \notin T_0$$

and we can define a strategy σ_t for player I exactly as we did in the proof of ($\flat$). Having in mind that we are assuming $s * \langle 0\rangle \notin T_0$ and $s * \langle 1\rangle \notin T_0$, it follows that $\sigma_t \otimes^{\mathrm{I}} \tau \in [T_1]$ iff $\sigma_t \otimes^{\mathrm{II}} \tau \notin [T_2]$ and so τ would not be a winning strategy for player II. This gives us a contradiction and therefore $s * \langle 0\rangle \in T_0 \vee s * \langle 1\rangle \in T_0$.

Assume $s * \langle 0\rangle \in T_0$ (we can deal with the case $s * \langle 1\rangle \in T_0$ in a similar way.) We must show that $s * \langle 0\rangle \in S_0 \vee s * \langle 1\rangle \in S_0$. To this end, define by primitive recursion $g : \mathbb{N} \to 2^{<\mathbb{N}}$ as follows:

$$g(0) = s * \langle 0\rangle, \quad g(i+1) = g(i) \otimes^{\mathrm{II}} \tau.$$

By ($\flat$), it easily follows that $\forall i\,(g(i) \in T_0)$. Since $\mathsf{Cyclic}(s)$ holds, there exists $d > 0$ such that $g(d) = s * \langle j\rangle$ for some $j < 2$. If $j = 0$ then $s * \langle 0\rangle = g(d-1) \otimes^{\mathrm{II}} \tau$. It then follows that $\mathsf{Cyclic}(s * \langle 0\rangle)$ holds and so $s * \langle 0\rangle \in S_0$. If $j = 1$ then $s * \langle 1\rangle \in T_0$ and we get that $g(2\cdot d) = s * \langle k\rangle$ for some $k < 2$. If $k = 0$ then we have $\mathsf{Cyclic}(s * \langle 0\rangle)$ and so $s * \langle 0\rangle \in S_0$; if $k = 1$ then we have $\mathsf{Cyclic}(s * \langle 1\rangle)$ and so $s * \langle 1\rangle \in S_0$. This completes the proof of the claim. □

Using Claim 3.11.2, we can easily obtain a path of T_0 as follows: Let us define $h : \mathbb{N} \to 2^{<\mathbb{N}}$ by putting

$$h(0) = \langle\rangle, \quad h(n+1) = \begin{cases} h(n) * \langle 0\rangle & \text{if } h(n) * \langle 0\rangle \in S_0 \\ h(n) * \langle 1\rangle & \text{if } h(n) * \langle 0\rangle \notin S_0 \end{cases}$$

By Δ^0_0-induction we get $\forall i\,(h(i) \in S_0)$ and therefore $f(i) = (h(i+1))(i)$ defines a path of T_0, which gives us a contradiction. As a consequence, player II cannot have a winning strategy either and, thus, the game $G_L([T_1], [T_2])$ would not be determined contradicting Δ^0_1-Det^*_L. □

In the next section we show that ACA_0 proves open (or closed) Lipschitz determinacy in the Cantor space. However, we close this section by showing that WKL_0 is still sufficient when one player plays in an open or closed set and her/his opponent plays in a clopen one (we will make use of this result in the proofs of Theorems 4.4 and 4.8.)

Proposition 3.12. WKL_0 *proves* (Δ^0_1, Σ^0_1)-Det^*_L *and* (Σ^0_1, Δ^0_1)-Det^*_L.

Proof. We work in an arbitrary model of WKL_0. We will prove (Δ^0_1, Π^0_1)-Det^*_L, which is equivalent to (Δ^0_1, Σ^0_1)-Det^*_L. (The proof for (Σ^0_1, Δ^0_1)-Det^*_L is similar and we omit it.) Consider $A(f) \in \Sigma^0_1$, $A'(f) \in \Pi^0_1$ and $B(g) \in \Pi^0_1$ such that $\forall f \in 2^{\mathbb{N}}\,(A(f) \leftrightarrow A'(f))$. We may safely assume that neither $A(f)$ nor $B(g)$ defines the empty set or the total set. By Proposition 3.1, there is a binary tree T such that $[T] = \{g \in 2^{\mathbb{N}} : B(g)\}$ and, by Lemma 3.6, there are nonempty pruned binary trees S, S' such that $[S] = \{f \in 2^{\mathbb{N}} : A(f)\}$ and $[S'] = \{f \in 2^{\mathbb{N}} : \neg A(f)\}$. As in Lemma 3.8, we consider $h_S = \max\{|s| : s \in S \cap S'\}$. Define $X = X_{in} \cap X_{out}$ where

$$X_{in} = \{t \in 2^{<\mathbb{N}} : |t| = h_S \wedge t \in T \wedge \exists g \in 2^{\mathbb{N}}\,(t \subset g \wedge g \in [T])\}, \text{ and}$$

$$X_{out} = \{t \in 2^{<\mathbb{N}} : |t| = h_S \wedge t \in T \wedge \exists t'\,(t \subseteq t' \wedge t' \notin T)\}$$

The existence of such sets follows by *bounded* Σ^0_1 or *bounded* Π^0_1 comprehension (which are well known to be provable from RCA_0) and by the fact that Π^0_1 formulas are closed in WKL_0 under quantifiers of the form $\exists g \in 2^{\mathbb{N}}$ (see Lemma 3.5). Intuitively, X comprises those positions of length h_S for which player II still has the possibility of playing inside or outside the closed set $[T]$.

<u>Case 1</u>: X is nonempty.

Then player II has a winning strategy. Namely, we define σ_{II} as follows. Pick $t_0 \in X$ and

g_{in} , $g_{out} \in 2^{\mathbb{N}}$ such that $t_0 \subset g_{in}$, $t_0 \subset g_{out}$, $g_{in} \in [T]$ and $g_{out} \notin [T]$. Given any sequence of odd length, $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n \rangle$, we define

$$\sigma_{\mathrm{II}}(s) = \begin{cases} t_0(n) & \text{if } n < h_S \\ g_{in}(n) & \text{if } n \geq h_S \text{ and } \langle x_0, \ldots, x_{h_S} \rangle \in S \\ g_{out}(n) & \text{if } n \geq h_S \text{ and } \langle x_0, \ldots, x_{h_S} \rangle \notin S \end{cases}$$

It is clear that σ_{II} exists by Δ^0_1 comprehension and in view of the properties of h_S in Lemma 3.8, it is immediate to see that σ_{II} is winning for player II.
Case 2: X is empty.
Then player I has a winning strategy. On the one hand, since $X = \emptyset$, we have $\neg \exists t\, (t \in X_{in} \wedge t \in X_{out})$ and thus

$$\forall t\, (t \in X_{in} \to \forall g \in 2^{\mathbb{N}}\, (t \subset g \to g \in [T])) \quad \text{and} \quad \forall t\, (t \in X_{out} \to \forall g \in 2^{\mathbb{N}}\, (t \subset g \to g \notin [T])).$$

On the other hand, picking $s_0 \in S \cap S'$ with $|s_0| = h_S$, it follows from Lemma 3.8, that there are f_{in} , $f_{out} \in 2^{\mathbb{N}}$ such that $s_0 \subset f_{in}$, $s_0 \subset f_{out}$, $f_{in} \in [S]$ and $f_{out} \notin [S]$. Having these facts in mind, given any sequence of even length, $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle$, we define

$$\sigma_{\mathrm{I}}(s) = \begin{cases} s_0(n) & \text{if } n < h_S \\ f_{in}(n) & \text{if } n \geq h_S \text{ and } \langle y_0, \ldots, y_{h_S - 1} \rangle \notin T \\ f_{out}(n) & \text{if } n \geq h_S \text{ and } \langle y_0, \ldots, y_{h_S - 1} \rangle \in X_{in} \\ f_{in}(n) & \text{if } n \geq h_S \text{ and } \langle y_0, \ldots, y_{h_S - 1} \rangle \in X_{out} \end{cases}$$

Again, σ_{I} exists by Δ^0_1 comprehension and it easy to see that σ_{I} is winning for player I. □

Let us observe that an alternative proof of Proposition 3.12 can be obtained by putting together Theorem 3.7 of [9] and the fact that a (Δ^0_1, Σ^0_1) Lipschitz game can be reduced to a $\mathrm{Bisep}(\Delta^0_1, \Sigma^0_1)$ infinite game, as defined in [9]. Also, note that in [4] it is proved that Proposition 3.12 can be extended a bit further: $(\Sigma^0_1 \wedge \Pi^0_1, \Delta^0_1)$-$\mathsf{Det}^*_L$ is still provable within WKL_0 (although we will not make use of this extended result in the present paper).

4. Determinacy for open sets and differences of closed sets

In the analysis of determinacy properties of a closed set $[T]$, the structure of its topological boundary will be relevant. In the previous section we have dealt with the simpler case where $[T]$ is a clopen set and, therefore, its boundary is empty. However, in cases where $[T]$ is not clopen ($[T]$ is, so to say, a "true" closed set) it must have some boundary points. The following definition isolates this notion.

Definition 4.1. We say that a binary tree T defines a true closed set if

$$\mathsf{TrueClosed}(T) \equiv \exists f \in 2^{\mathbb{N}}\, [f \in [T] \wedge \forall k\, \exists s\, (f[k] \subseteq s \wedge s \notin T)].$$

It is a well-known fact from general topology that a set is clopen if and only if its boundary is empty. Hence a basic dichotomy concerning closed sets emerges: either they are clopen or they must have some boundary points. ACA_0 is strong enough to show this fact:

Lemma 4.2. ACA_0 *proves that if T is a binary tree that does not define a clopen set then* $\mathsf{TrueClosed}(T)$.

Proof. Suppose that T is a binary tree that does not define a clopen set. Then, T does not properly define a finitely decidable set either and we have $\forall k\, \exists f \in 2^{\mathbb{N}}\, (f[k] \in T \wedge f \notin [T])$ and so

$$\forall k\, \exists s, t \in 2^{<\mathbb{N}}\, (|s| = k \wedge s \subset t \wedge s \in T \wedge t \notin T). \tag{†}$$

Define T' to be $\{s \in 2^{<\mathbb{N}} : s \in T \wedge \exists t\, (s \subset t \wedge t \notin T)\}$. Note that T' exists by Σ^0_1 comprehension (which is available thanks to ACA_0). Clearly, T' is a binary tree and it follows by (†) that T' is infinite. By applying Weak König Lemma we obtain that T' has a path, say $g \in 2^{\mathbb{N}}$. Since $T' \subseteq T$, $g \in [T]$. In addition, by the definition of T' we have $\forall k\, \exists s\, (g[k] \subset s \wedge s \notin T)$. Thus, we have shown that $\mathsf{TrueClosed}(T)$ holds, as required. □

Note that in the proof of Lemma 4.2 we have indeed shown the following dichotomy property to hold:

Definition 4.3. Let (DP) denote the formula

$$\forall T\, (\mathsf{BinaryTree}(T) \rightarrow \mathsf{TrueClosed}(T) \vee T \text{ properly defines a finitely decidable set}),$$

where $\mathsf{BinaryTree}(T)$ is a formula declaring that T is a binary tree.

We shall use (DP) as a basic principle to be added to RCA_0 in order to derive Σ^0_1-determinacy.

Theorem 4.4. *The following principles are provable in* $\mathsf{RCA}_0 + (\mathsf{DP})$*:*

1. Σ^0_1-Det^*_L.
2. Σ^0_1-Det^*_W.

Proof. Firstly, let us note that $\mathsf{RCA}_0 + (\mathsf{DP})$ extends WKL_0. Indeed, if T is an infinite binary tree and $\mathsf{TrueClosed}(T)$ holds then, by definition, T has a path. If $\mathsf{TrueClosed}(T)$ does not hold, by (DP) there is some k such that for all $f \in 2^{\mathbb{N}}$, $f \in [T] \leftrightarrow f[k] \in T$. Since T is infinite, there exists $s \in T$ with $|s| > k$ and we define $g \in 2^{\mathbb{N}}$ by putting $g(j) = s(j)$, for $j < |s|$, and $g(j) = 0$ for $j \geq |s|$. It is clear that $g \in [T]$, as required.

Hence, we work in an arbitrary model of $\mathsf{WKL}_0 + (\mathsf{DP})$.

(1) We will prove Π^0_1-Det^*_L, which is equivalent to Σ^0_1-Det^*_L. Consider $A(f), B(g) \in \Pi^0_1$. By Proposition 3.1 there are binary trees S and T satisfying that $[S] = \{f \in 2^{\mathbb{N}} : A(f)\}$ and $[T] = \{g \in 2^{\mathbb{N}} : B(g)\}$. We must show that the game $G_L([S], [T])$ is determined.

Case A: $\mathsf{TrueClosed}(T)$ holds.

Then, player II has a winning strategy. Actually, pick $g_0 \in 2^{\mathbb{N}}$ satisfying that $g_0 \in [T] \wedge \forall k\, \exists t\, (g_0[k] \subset t \wedge t \notin T)$. Using RCA_0, we get $h : \mathbb{N} \to 2^{<\mathbb{N}}$ such that $\forall k\, (g_0[k] \subset h(k) \wedge h(k) \notin T)$. Define H to be the set given by

$$(k, n, i) \in H \leftrightarrow (n < |h(k)| \wedge i = h(n)) \vee (n \geq |h(k)| \wedge i = 0).$$

Clearly, H exists by Δ^0_1 comprehension. We will write $H_k(n) = i$ for $(k, n, i) \in H$. Thus, each function H_k extends the finite sequence $h(k)$ by putting zeros on the end. We are now in a position to define a strategy for player II, σ_{II}, as follows. Given any sequence of odd length $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n \rangle$, we define

$$\sigma_{\mathrm{II}}(s) = \begin{cases} g_0(n) & \text{if } \langle x_0, \ldots, x_n \rangle \in S \\ H_k(n) & \text{if } \langle x_0, \ldots, x_n \rangle \notin S \text{ and } k = \mu j\ (\langle x_0, \ldots, x_j \rangle \notin S) \end{cases}$$

(In words, player II plays using the boundary point g_0 while player I has played inside S and if player I leaves S at round k then player II will also leave T by using $h(k)$.) Again, σ_{II} exists by Δ^0_1 comprehension and it is straightforward to see that σ_{II} is winning for player II.

Case B: $\mathsf{TrueClosed}(T)$ does not hold.
By (DP) there exists some $k_0 \in \mathbb{N}$ satisfying that $\forall g \in 2^{\mathbb{N}}\,(g \in [T] \leftrightarrow g[k_0] \in T)$. Let $B'(g)$ be the Σ^0_1-formula $g[k_0] \in T$. Then, $\forall g \in 2^{\mathbb{N}}\,(B(g) \leftrightarrow B'(g))$ and so $G_L([S],[T])$ is determined by Proposition 3.12.

(2) We will prove again Π^0_1-$\mathsf{Det}^{\star}_W$. Consider $A(f)$, $B(g) \in \Pi^0_1$. By Proposition 3.1 there are binary trees S and T satisfying that $[S] = \{f \in 2^{\mathbb{N}} : A(f)\}$ and $[T] = \{g \in 2^{\mathbb{N}} : B(g)\}$. We must show that the game $G_W([S],[T])$ is determined. The proof is similar to that of the Lipschitz case. Since a winning strategy for player II in $G_L([S],[T])$ immediately gives rise to a winning strategy for player II in $G_W([S],[T])$, the only situation that deserves some explanations is case B above. Thus, assume that $\mathsf{TrueClosed}(T)$ does not hold. If $\mathsf{TrueClosed}(S)$ does not hold either, by (DP) both T and S define a clopen set and $G_W([S],[T])$ is determined by Theorem 3.9. So, let us assume that $\mathsf{TrueClosed}(S)$ holds.

On the one hand, there exists $f_0 \in 2^{\mathbb{N}}$ such that $f_0 \in [S] \wedge \forall k\, \exists s\, (f_0[k] \subseteq s \wedge s \notin S)$ and, on the other hand, by (DP) there exists some $k_0 \in \mathbb{N}$ such that $\forall g \in 2^{\mathbb{N}}\,(g \in [T] \leftrightarrow g[k_0] \in T)$. Using RCA_0, we get $h : \mathbb{N} \to 2^{<\mathbb{N}}$ satisfying that $\forall k\,(f_0[k] \subseteq h(k) \wedge h(k) \notin S)$. As in the proof of Case A of part (1), there exists a sequence of functions, $\{H_k : k \in \mathbb{N}\}$, such that each H_k extends the finite sequence $h(k)$ by putting zeros on the end. Since now player II is allowed to pass, we also need a function $ext : 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that $ext(s)$ is the finite sequence obtained by dropping the zeros of the finite sequence s and decreasing the values by 1 (Recall that in coding a strategy for a Wadge game, for player II we identify passing with picking the number 0 and for both players we identify playing i with picking $i+1$.) We are now in a position to define a winning strategy for player I. Given any sequence of even length, $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle$, we put

$$\sigma_{\mathrm{I}}(s) = \begin{cases} f_0(n)+1 & \text{if } |ext(\langle y_0, \ldots, y_{n-1} \rangle)| < k_0 \\ H_k(n)+1 & \begin{array}{l} \text{if } |ext(\langle y_0, \ldots, y_{n-1} \rangle)| \geq k_0 \text{ and } ext(\langle y_0, \ldots, y_{n-1} \rangle)[k_0] \in T \\ \text{and } k = \mu j\,(|ext(\langle y_0, \ldots, y_{j-1} \rangle)| \geq k_0) \end{array} \\ f_0(n)+1 & \text{if } |ext(\langle y_0, \ldots, y_{n-1} \rangle)| \geq k_0 \text{ and } ext(\langle y_0, \ldots, y_{n-1} \rangle)[k_0] \notin T \end{cases}$$

(In words, player I plays using the boundary point f_0 until player II has played (not passed) k_0 times. At that round player II has already decided whether or not she will play inside T and player II will play accordingly.) Then, σ_{I} exists by Δ^0_1 comprehension and it is easy to verify that σ_{I} is winning for player I. This completes the proof of the theorem. □

Corollary 4.5. *Σ^0_1-Det^*_L and Σ^0_1-Det^*_W are provable in ACA_0.*

Proof. It follows from Lemma 4.2 and Theorem 4.4. □

Remark 4.6. The fact that ACA_0 proves Σ^0_1-Det^*_L can also be derived from known results on determinacy of infinite games. On the one hand, in [11] Nemoto, MedSalem and Tanaka showed that ACA_0 proves general determinacy for the second level of the difference hierarchy $(\Sigma^0_1)_2$ (where $(\Sigma^0_1)_2$ coincides with $\Sigma^0_1 \wedge \Pi^0_1$). On the other hand, we showed in Remark 2.4 that an open Lipschitz game can be reduced to a infinite game of pay-off set complexity $\Sigma^0_1 \wedge \Pi^0_1$.

In the next result we calibrate the exact strength of the principle (DP) over RCA_0. The basic ideas in the proof were suggested to us by Paul Shafer. As a matter of fact, he has proved a stronger version of this result.[1]

Theorem 4.7. *The following principles are equivalent over* RCA_0*:*

1. ACA_0.
2. (DP).
3. *Each infinite binary tree has a leftmost path. That is, for every infinite binary tree T there is $f \in [T]$ such that for any other path $g \in [T]$, there exists $k \in \mathbb{N}$ such that $f[k] = g[k]$ and $f(k) < g(k)$.*

Proof. (1) $\Rightarrow$ (2) follows by Lemma 4.2.
(2) $\Rightarrow$ (3): We work in $\mathsf{WKL}_0 + (\mathsf{DP})$ (recall that in the proof of Theorem 4.4 we showed that $\mathsf{RCA}_0 + (\mathsf{DP})$ extends WKL_0.) Let T be an infinite binary tree. We shall prove that T has a leftmost path. To this end, let T^* be the set of binary finite sequences defined by

$$s \in T^* \quad \Longleftrightarrow \quad \begin{cases} s \in T \vee \\ \exists s_0\, \exists t \begin{cases} s = s_0 * t \wedge s_0 \in T \wedge s_0 * \langle t(0)\rangle \notin T \wedge \\ \exists s_1 \in T\, (|s_1| = |s| \wedge \mathsf{left}(s_1, s_0)) \end{cases} \end{cases}$$

where $\mathsf{left}(s_1, s_0)$ expresses that s_1 is to the left of s_0. Namely,

$$s_1 \subseteq s_0 \vee \exists j < |s_1|\, (s_1[j] = s_0[j] \wedge s_1(j) < s_0(j)).$$

Likewise, for $f, g \in 2^{\mathbb{N}}$ we say that f is to the left of g if $\exists j\, (f[j] = g[j] \wedge f(j) < g(j))$. It can be easily checked, by using Δ^0_1-comprehension, that T^* does exist and that it is a tree. The following facts easily follow from the definition of T^*:

(†) Suppose $g \in [T^*]$. If $g \notin [T]$ then there is $f \in [T]$ such that f is to the left of g.

Proof. If $g \notin [T]$ then there exists $l \in \mathbb{N}$ such that $g[l+1] \notin T \wedge g[l] \in T$. For each $m > l$, since $g[m] \in T^* - T$, it follows from the definition of T^* that there is $s_m \in T$ satisfying $\mathsf{left}(s_m, g[l])$ and $|s_m| = |g[m]| = m$. As a consequence, we get an infinite tree $T_{l,g} = \{s \in T : |s| \leq l \vee \mathsf{left}(s, g[l])\}$. By WKL_0, there exists $f \in [T_{l,g}] \subseteq [T]$ and it is obvious that f is to the left of g. □

(‡) Suppose $g \in [T^*]$. If there is $f \in [T]$ such that f is to the left of g, then g is an interior point of $[T^*]$ (that is to say, $\exists k\, \forall s \in 2^{<\mathbb{N}}\, (g[k] \subseteq s \rightarrow s \in T^*)$.)

[1] Paul Shafer has proved ([12], personal communication) that ACA_0 can be already derived over RCA_0 from the principle: "Every closed and not open set in the Cantor space has a boundary point."

Proof. Let $f \in [T]$ be such that for some k, $f[k] = g[k]$ and $f(k) < g(k)$. For all $m > k$, we have

$$f[m] \in T \wedge \mathsf{left}(f[m], g[k+1]).$$

It easy to see that for all $w \in 2^{<\mathbb{N}}$, $s = g[k+1] * w \in T^*$. To check this, observe that if $s \notin T$ then there is $l \geq k$ such that $s[l] \in T$ and $s[l] * \langle s(l) \rangle \notin T$. Therefore, taking $s_0 = s[l]$ and $t \in 2^{<\mathbb{N}}$ such that $s = s_0 * t$, we get that $\mathsf{left}(f[|s|], s_0)$ holds (note that $f[k] \subseteq s_0$ and $f(k) < s(k)$) and so $s \in T^*$. □

Since $T \subseteq T^*$, T^* is infinite, and by (DP) either T^* properly defines a finitely decidable set or $\mathsf{TrueClosed}(T^*)$ holds. We distinguish these two cases.

<u>Case 1</u>: T^* properly defines a finitely decidable set.
By Lemma 3.6 we can assume without loss of generality that T^* is a pruned tree. Let $g \in [T^*]$ defined by

$$g(n) = \min\{j \in \{0, 1\} : g[n] * \langle j \rangle \in T^*\}.$$

Clearly, g is the leftmost path of T^*. By (†), $g \in [T]$ and, since $[T] \subseteq [T^*]$, g is also the leftmost path of $[T]$.

<u>Case 2</u>: $\mathsf{TrueClosed}(T^*)$ holds.
Then there exists $g \in [T^*]$ such $\forall j\, \exists s\, (g[j] \subseteq s \wedge s \notin T^*)$. Observe that by (†), if $g \notin [T]$ then there is $f \in [T]$ such that f is to the left of g. But then by (‡), g would be an interior point of $[T^*]$, contradicting our hypothesis on g. Thus, $g \in [T]$. As a consequence, g is the leftmost path of T (for otherwise again by (‡), g would be an interior point of $[T^*]$.)
(3) ⇒ (1): See [3], Lemma 3.1. □

Using the same ideas as in the proof of Theorem 4.4, we can derive a slightly sharper version of Corollary 4.5, which will be useful in section 5 to characterize ACA_0 in terms of the Semi-Linear Ordering Principle SLO.

Theorem 4.8.

(1) ACA_0 *proves* $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*_L$.
(2) ACA_0 *proves* $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*_W$.

Proof. We work in an arbitrary model of ACA_0.
(1) Consider $A(f) \in \Sigma^0_1$ and $B(g) \in \Sigma^0_1 \wedge \Pi^0_1$. We must show that $G_L(A, B)$ is determined. By Lemma 3.6, there exist binary pruned trees S, T_0 and T_1 such that $T_1 \subseteq T_0$ and

$$A(f) \leftrightarrow f \notin [S], \text{ and } B(g) \leftrightarrow g \in [T_0] - [T_1].$$

Put $T_2 = \{t \in T_1 : \exists t'(t' \in T_0 - T_1 \wedge t \subseteq t')\}$. Clearly, T_2 is a subtree of T_1 and T_2 exists by Σ^0_1-comprehension.
<u>Case A</u>: $[T_2] \neq \emptyset$.
Then, player II has a winning strategy in the game $G_L(A, B)$. Actually, pick $g_0 \in [T_2]$. Then g_0 satisfies that

$$g_0 \in [T_1] \wedge \forall k\, \exists t \in T_0\, (g_0[k] \subset t \wedge t \notin T_1)].$$

By Δ^0_1 comprehension, there exists $h : \mathbb{N} \to T_0$ satisfying that $\forall k\, (g_0[k] \subset h(k) \wedge h(k) \in T_0 \wedge h(k) \notin T_1)$. Define $H : \mathbb{N}^2 \to \mathbb{N}$ to be the function defined by recursion as follows:

$$H(k,n) = \begin{cases} g_0(n) & \text{if } n < k \\ h(k)(n) & \text{if } k \le n < |h(k)| \\ \min\{j \le 1 : \langle H(k,0), \ldots, H(k,n-1), j\rangle \in T_0\} & \text{otherwise} \end{cases}$$

Since T_0 is a pruned tree, H is well defined and, writing $H_k(n) = H(k,n)$, each function H_k extends the finite sequence $h(k)$ to a path through $[T_0]$. We are now in a position to define a strategy for player II, σ_{II}, as follows. Given any sequence of odd length $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n\rangle$, we define

$$\sigma_{\mathrm{II}}(s) = \begin{cases} g_0(n) & \text{if } \langle x_0,\, x_1, \ldots, x_n\rangle \in S \\ H_k(n) & \text{if } \langle x_0,\, x_1, \ldots, x_n\rangle \notin S \text{ and } k = \min\{j : \langle x_0,\, x_1, \ldots, x_j\rangle \notin S\} \end{cases}$$

Again, σ_{II} exists by Δ^0_1 comprehension and it is straightforward to see that σ_{II} is winning for player II.

Case B: $[T_2] = \emptyset$.
By WKL_0, T_2 must be finite. Pick $k_0 \in \mathbb{N}$ such that all sequences in T_2 have length below k_0. Consequently, we obtain that

$$\forall g \in 2^{\mathbb{N}} (B(g) \leftrightarrow g \in [T_0] \wedge g[k_0] \notin T_1)$$

and hence $B(g)$ is equivalent to a Π^0_1-formula. Let T be a binary pruned tree such that $[T] = \{g \in 2^{\mathbb{N}} : B(g)\}$. We must show that $G_L(A,[T])$ is determined. We distinguish two cases.

Case B.1: $\mathsf{TrueClosed}(S)$ holds.
Then, player I has a winning strategy. To see this, pick $f_0 \in [S]$ such that $\forall k\, \exists s\, (f_0[k] \subseteq s \wedge s \notin S)$. By Δ^0_1 comprehension, there exists $h : \mathbb{N} \to 2^{<\mathbb{N}}$ such that $\forall k\, (f_0[k] \subseteq h(k) \wedge h(k) \notin S)$ and so we have

$$\neg A(f_0) \wedge \forall k\, \forall f \in 2^{\mathbb{N}}\, (h(k) \subset f \to A(f)). \tag{†}$$

As in the previous case, consider a sequence of functions, $\{H_k : k \in \mathbb{N}\}$, such that each function H_k extends the finite sequence $h(k)$ but now by putting zeros on the end. We define a strategy for player I as follows. Given any sequence of even length $s = \langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle$, we put

$$\sigma_{\mathrm{I}}(s) = \begin{cases} f_0(n) & \text{if } \langle y_0, \ldots, y_{n-1}\rangle \in T \\ H_k(n) & \text{if } \langle y_0, \ldots, y_{n-1}\rangle \notin T \text{ and } k = \mu j\ (\langle y_0, \ldots, y_{j-1}\rangle \notin T) \end{cases}$$

Note that σ_{I} exists by Δ^0_1 comprehension and it follows by (†) that σ_{I} is winning for player I.

Case B.2: $\mathsf{TrueClosed}(S)$ does not hold.
By (DP), there is $k_1 \in \mathbb{N}$ such that $\forall f \in 2^{\mathbb{N}}\, (A(f) \leftrightarrow f[k_1] \notin S)$. Thus, the fact $G_L(A,[T])$ is determined follows from (Δ^0_1, Π^0_1)-$\mathsf{Det}^\star_L$ (which is available in WKL_0 by Proposition 3.12).

(2) The proof of part (1) can be easily adapted to provide a proof of $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*_W$. We omit the details. □

Remark 4.9. In [5], using similar but combinatorially more complex methods, it was showed that ACA_0 also proves $(\Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*_L$. As a matter of fact a reversal for ACA_0 was derived by showing that, over RCA_0, this determinacy principle is equivalent to ACA_0. However, in the present article Theorem 4.8 suffices to obtain reversals for ACA_0 (see Theorems 5.5 and 5.7 below).

5. The Semi-Linear Ordering Principle

In the setting of descriptive set theory and working in the Baire space, in [17] Wadge showed that the relations $\leq_L$ (reducibility via Lipschitz functions) and $\leq_W$ (reducibility via continuous functions) can be characterized in terms of infinite games. Wadge's Lemma from [17] states that *i)* player II has a winning strategy in $G_L(A,B)$ (resp. $G_W(A,B)$) iff $A \leq_L B$ (resp. $A \leq_W B$); and *ii)* if player I has a winning strategy in $G_L(A,B)$ or $G_W(A,B)$ then $B^c \leq_L A$, where X^c denotes the complement of the set X. The Axiom of Determinacy AD therefore implies the following comparability property (known as the *Semi-Linear Ordering principle* SLO):

$$\mathrm{SLO}^W = \text{“For all } A, B \subseteq \omega^\omega\text{, either } A \leq_W B \text{ or } B^c \leq_W A\text{,”}$$
$$\mathrm{SLO}^L = \text{“For all } A, B \subseteq \omega^\omega\text{, either } A \leq_L B \text{ or } B^c \leq_L A\text{.”}$$

Wadge soon realized the relevance of SLO in order to establish the structure of the Lipschitz/Wadge degrees (i.e. the equivalence classes generated by the pre-orders $\leq_L$ and $\leq_W$). He also showed that SLO shares important consequences with AD: SLO^W proves the perfect set property (every subset of ω^ω is either countable or else it contains a copy of the Cantor set) and therefore SLO^W is incompatible with the Axiom of Choice. In fact, some years later, A. Andretta (see [1] and [2]) was able to prove that, over certain set-theoretic base theory, the principles AD^W="all Wadge games are determined," AD^L="all Lipschitz games are determined," SLO^L and SLO^W are pairwise equivalent. (Whether SLO^W is also equivalent to the full Axiom of Determinacy AD under some appropriate set-theoretical assumptions is still an open question.)

In this section we present a formalization of the semi-linear ordering principle within second order arithmetic and we initiate the study of the reverse mathematics of this principle. Our formalization is based on the aforementioned characterization of $\leq_L$ and $\leq_W$ in terms of Lipschitz/Wadge games. Fix $X \subseteq \mathbb{N}$ nonempty and consider formulas $A(f)$ and $B(g)$ with distinguished function variables $f, g \in X^{\mathbb{N}}$. We say that A *is Lipschitz reducible to* B if player II has a winning strategy in the Lipschitz game $G_L^X(A,B)$:

$$\mathsf{Red}_L^X(A,B) \equiv \exists \sigma_{\mathrm{II}} \, \forall \sigma_{\mathrm{I}} \, (A(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_L^{\mathrm{II}} \sigma_{\mathrm{II}})),$$

where σ_{I} and σ_{II} range over strategies for players I and II, respectively. In a similar vein, A *is Wadge reducible to* B if player II has a winning strategy in the Wadge game $G_W^X(A,B)$:

$$\mathsf{Red}_W^X(A,B) \equiv \exists \sigma_{\mathrm{II}} \, \forall \sigma_{\mathrm{I}} \, [\mathsf{Inf}(\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}) \wedge (A(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{I}} \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma_{\mathrm{I}} \otimes_W^{\mathrm{II}} \sigma_{\mathrm{II}}))].$$

Definition 5.1. Fix $X = \{0,1\}$ or $\mathbb{N}$. Let Γ_1 and Γ_2 be classes of formulas with distinguished function variables $f, g \in X^{\mathbb{N}}$, respectively.

1. The scheme of (Γ_1, Γ_2) Lipschitz semi-linear ordering principle in $X^{\mathbb{N}}$, denoted (Γ_1,Γ_2)-SLO_L^X, is given by the axiom scheme $\mathsf{Red}_L^X(A,B) \vee \mathsf{Red}_L^X(\neg B, A)$, where $A(f)$ is in Γ_1 and $B(g)$ is in Γ_2.
2. The scheme of (Γ_1, Γ_2) Wadge semi-linear ordering principle in $X^{\mathbb{N}}$, denoted (Γ_1,Γ_2)-SLO_W^X, is given by the axiom scheme $\mathsf{Red}_W^X(A,B) \vee \mathsf{Red}_W^X(\neg B, A)$, where $A(f)$ is in Γ_1 and $B(g)$ is in Γ_2.
3. (Δ_n^0, Δ_m^0)-$\mathsf{SLO}_{L/W}^X$, (Γ, Δ_n^0)-$\mathsf{SLO}_{L/W}^X$ and (Δ_n^0, Γ)-$\mathsf{SLO}_{L/W}^X$ are defined similarly.

Using our notation conventions, we will simply write Γ-$\mathsf{SLO}^X_{L/W}$ or Δ^0_n-$\mathsf{SLO}^X_{L/W}$ when $\Gamma_1 = \Gamma_2 = \Gamma$ or $n = m$. Also, we will omit the superscript $\mathbb{N}$ for schemes in the Baire space, and we replace $\{0,1\}$ with $*$ for schemes in the Cantor space.

Next lemma states two basic properties of the semi-linear ordering principle: SLO can be inferred from determinacy, and SLO_L implies SLO_W.

Lemma 5.2. *Fix $X = \{0,1\}$ or $\mathbb{N}$. It is provable over RCA_0 that*

1. (Γ_1,Γ_2)-$\mathsf{Det}^X_{L/W}$ *implies* (Γ_1,Γ_2)-$\mathsf{SLO}^X_{L/W}$*, and the same holds for classes* (Δ^0_n,Δ^0_m), (Γ,Δ^0_m), (Δ^0_n,Γ).
2. (Γ_1,Γ_2)-SLO^X_L *implies* (Γ_1,Γ_2)-SLO^X_W*, and the same holds for classes* (Δ^0_n,Δ^0_m), (Γ,Δ^0_m), (Δ^0_n,Γ).

Proof. (1): We only write the proof for the Lipschitz case, the Wadge case being analogous. Reasoning in RCA_0, assume (Γ_1,Γ_2)-Det^X_L. Consider $A(f) \in \Gamma_1$ and $B(g) \in \Gamma_2$. We must show that either $\mathsf{Red}^X_L(A,B)$ or $\mathsf{Red}^X_L(\neg B,A)$ holds. By hypothesis, $G^X_L(A,B)$ is determined. If player II has a winning strategy in that game then there is nothing to prove. So assume that player I has a winning strategy, i.e. there is σ such that $\forall\sigma_{\mathrm{II}}\, \neg(A(\sigma \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}}) \leftrightarrow B(\sigma \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}}))$ holds. Define τ to be the strategy for player II given by

$$\tau(\langle x_0\rangle) = \sigma(\langle\rangle),$$
$$\tau(\langle x_0, y_0, \ldots, x_k, y_k, x_{k+1}\rangle) = \sigma(\langle y_0, x_0, \ldots, y_k, x_k\rangle).$$

It is easy to check that τ is a winning strategy for player II in $G^X_L(\neg B, A)$, as required.

(2): It suffices to note that a winning strategy for player II in $G^X_L(A,B)$ automatically gives rise to a winning strategy for player II in the corresponding Wadge game $G^X_W(A,B)$. $\square$

In view of Lemma 5.2 and our results on L/W-determinacy in the previous sections, we obtain that

Corollary 5.3.

1. RCA_0 *proves* Δ^0_1-SLO^*_W, (Δ^0_1,Σ^0_n)-SLO^*_W *and* (Σ^0_n,Δ^0_1)-SLO^*_W*, for every $n > 0$.*
2. WKL_0 *proves* Δ^0_1-SLO^*_L, (Δ^0_1,Σ^0_1)-SLO^*_L *and* (Σ^0_1,Δ^0_1)-SLO^*_L.
3. ACA_0 *proves* Σ^0_1-$\mathsf{SLO}^*_{L/W}$ *and* $(\Sigma^0_1,\Sigma^0_1\wedge\Pi^0_1)$-$\mathsf{SLO}^*_{L/W}$.

Proof. Only the fact that RCA_0 implies (Σ^0_n,Δ^0_1)-SLO^*_W is not a direct consequence of previously proved results for L/W-determinacy. Consider $n > 0$, $A(f) \in \Sigma^0_n$ and $B(g) \in \Delta^0_1$. Then, $\neg B(g)$ is also in Δ^0_1 and we get that $\mathsf{Red}^X_W(\neg B, A)$ holds by reasoning as in part (2) of Proposition 3.9. $\square$

The main results of the present section are two reversals for ACA_0 in terms of the semi-linear ordering principle SLO^*_W (Propositions 5.4 and 5.6 below). The interest of these results is, we think, twofold. First, since SLO^*_W is the weakest one among the considered axiom schemes, obtaining reversals in terms of SLO^*_W makes our results stronger. Second, as a by product we establish that, for Σ^0_1 and $(\Sigma^0_1,\Sigma^0_1\wedge\Pi^0_1)$ sets SLO^*_W is, after all, as strong as SLO^*_L and L/W-determinacy (a miniaturisation of the above mentioned Andretta's result in Set Theory.)

Proposition 5.4. *Over* WKL_0, Σ^0_1-SLO^*_W *implies* ACA_0.

Proof. By Theorem 4.7 it is sufficient to show that $\mathsf{WKL}_0 + \neg(\mathsf{DP})$ implies that Π^0_1-SLO^*_W fails. Work in WKL_0 and assume $\neg(\mathsf{DP})$. Then there is a binary tree T such that T does not properly define a finitely decidable set and $\mathsf{TrueClosed}(T)$ does not hold. Hence, we have

i) $\neg\exists k\,\forall f \in 2^{\mathbb{N}}\,(f[k] \in T \to f \in [T])$, and

ii) $\forall f \in 2^{\mathbb{N}}\,(f \in [T] \to \exists k\,\forall t\,(f[k] \subseteq t \to t \in T))$.

Define $A(f) \equiv \forall i\,(f(i) = 0)$ and $B(g) \equiv g \in [T]$. Note that $A, B \in \Pi^0_1$.

Claim 5.4.1. Player II cannot have a winning strategy in $G_W(A, B)$.

Proof. Towards a contradiction, suppose σ is a winning strategy for player II in that game. Consider the strategy for player I given by $\tau_1(s) = 0$ for all $s \in \mathrm{Seq}^{\{0,1,2\}}_{\mathrm{even}}$ (We will use the terminology and conventions for formalized Wadge games introduced in subsection 2.2.) Then we must have $\mathsf{Inf}(\tau_1, \sigma)$ and $h = \tau_1 \otimes^{\mathrm{II}}_W \sigma \in [T]$. By condition ii) above, there exists $k_0 \in \mathbb{N}$ such that $\forall t\,(h[k_0] \subseteq t \to t \in T))$. Consider

$$\begin{aligned} move(0) &= \mu i\,[(\tau_1 \otimes \sigma)(2i+1) \neq 0] \\ move(n+1) &= \mu i\,[i > move(n) \wedge (\tau_1 \otimes \sigma)(2i+1) \neq 0] \end{aligned}$$

and take $k_1 = move(k_0 - 1)$. That is to say, player II has already played (not passed) k_0 times after her first $k_1 + 1$ moves. Define a new strategy for player I, τ_2, by putting $\tau_2(s) = 0$ if $|s| \leq 2 \cdot k_1$ and $\tau_2(s) = 2$ otherwise. In words, player I plays as in τ_1 during his first $k_1 + 1$ moves and then he leaves the set A. Clearly, we have $\neg A(\tau_2 \otimes^{\mathrm{I}}_W \sigma)$ and $B(\tau_2 \otimes^{\mathrm{II}}_W \sigma)$. But this contradicts the fact that σ is winning for player II. □

Claim 5.4.2. Player II cannot have a winning strategy in $G_W(\neg B, A)$.

Proof. Towards a contradiction, suppose σ is a winning strategy for player II in that game. As in the proof of Theorem 3.11, $s \otimes^{\mathrm{II}} \sigma$ denotes the finite sequence consisting of player II's moves (including 0 for representing passing) when she uses σ and player I plays according to s in his first moves. Put

$$S = \{s : s \in T \wedge \forall i < |s|\,((s \otimes^{\mathrm{II}} \sigma)(i) \neq 2)\}.$$

In words, S comprises those positions s in T for which player II either passes or chooses 0 if she uses the strategy σ and player I plays according to s. Clearly, S exists by Δ^0_1-comprehension and S is a binary tree contained in T.

Let us see that S must be infinite. Pick $k_1 \in \mathbb{N}$. By condition i) above, there exists $s_1 \in T$ and $i_0 \in \{0, 1\}$ such that $|s_1| = k_2 > k_1$ and $s_1 * \langle i_0 \rangle \notin T$. Then, we must have $\forall i < k_2\,(s_1 \otimes^{\mathrm{II}} \sigma)(i) \neq 2$. For assume the contrary and suppose that there is $j < k_2$ such that $(s_1 \otimes^{\mathrm{II}} \sigma)(j) = 2$. Define a strategy for player I, τ_1, by putting $\tau_1(s) = s_1(k)$ if $2 \cdot k = |s| < k_2$ and $\tau_1(s) = i_0$ otherwise. (In words, player I uses s_1 in his first k_2 moves and then he leaves the set B.) It is clear that $\neg B(\tau_1 \otimes^{\mathrm{I}}_W \sigma)$ and $\neg A(\tau_1 \otimes^{\mathrm{II}}_W \sigma)$ hold, which is impossible since σ is a winning strategy for player II. Therefore, s_1 is in S and so S contains arbitrarily long sequences, as required.

Since S is infinite, by using WKL_0 we get a path of S, say $h \in 2^{\mathbb{N}}$. Consider now the strategy for player I, τ_2, given by $\tau_2(s) = h(k)$ if $|s| = 2 \cdot k$. Then we have $B(\tau_2 \otimes^{\mathrm{I}}_W \sigma)$ and $A(\tau_2 \otimes^{\mathrm{II}}_W \sigma)$, which again contradicts the fact that σ is a winning strategy for player II. □

It follows from Claims 5.4.1 and 5.4.2 that Π^0_1-SLO^*_W fails, as required. □

Putting together Proposition 5.4 and our previous results, we obtain that

Theorem 5.5. *Over* RCA_0*, the following principles are pairwise equivalent.*

1. ACA_0.
2. Σ^0_1-Det^*_L.
3. $\mathsf{WKL}_0 + \Sigma^0_1$-$\mathsf{Det}^*_W$.
4. $\mathsf{WKL}_0 + \Sigma^0_1$-$\mathsf{SLO}^*_{L/W}$.

Proof. Firstly, (1) $\Rightarrow$ (2) follows from Corollary 4.5; (2) $\Rightarrow$ (4) follows from Theorem 3.11 and Lemma 5.2; and (4) $\Rightarrow$ (1) follows from Proposition 5.4 and Lemma 5.2. Secondly, (1) $\Rightarrow$ (3) follows from Corollary 4.5; and (3) $\Rightarrow$ (1) follows from Proposition 5.4 and Lemma 5.2. ☐

It is natural to ask ourselves whether WKL_0 can be eliminated from items (3) and (4) of Theorem 5.5. In section 3 we showed Δ^0_1-Det^*_W (and, as a consequence, Δ^0_1-SLO^*_W) to be rather weak (they are provable from our base theory RCA_0) but the question of calibrating the exact strength of Δ^0_1-SLO^*_L over RCA_0 has been left pending. Note that if we were able to prove Δ^0_1-SLO^*_L and WKL_0 to be equivalent then it would follow from Proposition 5.4 that ACA_0 and Σ^0_1-SLO^*_L are equivalent over the base theory RCA_0.

In fact, our second reversal will provide us with a characterization of ACA_0 in terms of SLO over plain RCA_0, but it comes at a price, as we have to increase the quantifier complexity of player II's pay-off set.

Proposition 5.6. *Over* RCA_0*,* $(\Sigma^0_1, \Sigma^0_1\wedge\Pi^0_1)$-$\mathsf{SLO}^*_W$ *implies* ACA_0.

Proof. Assume $\mathsf{RCA}_0 + (\Sigma^0_1, \Sigma^0_1\wedge\Pi^0_1)$-$\mathsf{SLO}^*_L$ and take $\varphi(x) \in \Sigma^0_1$ (we disregard parameters). We must show that the set $\{x : \varphi(x)\}$ exists. Define $A(f)$ to be $\exists k\,(f(k) = 1 \wedge \forall k' < k\,(f(k') = 0))$ and define $B(g)$ to be

$$\exists k\,(g(k) = 1 \wedge \forall k' < k\,(g(k') = 0) \wedge \forall i \le k\,(g(k+i+1) = 1 \to \varphi(i))) \wedge$$
$$\forall k\,(g(k) = 1 \wedge \forall k' < k\,(g(k') = 0) \to \forall i \le k\,(\varphi(i) \to g(k+i+1) = 1)).$$

That is to say, a play for player I is in A if it is of the form $0^{(k)} * \langle 1\rangle * f'$ for some $k \in \mathbb{N}$ and $f' \in 2^{\mathbb{N}}$, whereas a play for player II is in B if it is of the form $0^{(l)} * \langle 1\rangle * \langle t_0, t_1, \ldots, t_l\rangle * g'$ for some $g' \in 2^{\mathbb{N}}$ and, in addition, for each $i \le l$, $t_i = 1$ iff $\varphi(i)$ holds. It is clear that A is in Σ^0_1 and B is in $\Sigma^0_1\wedge\Pi^0_1$.

Claim 5.6.1. Player II cannot have a winning strategy in the game $G_W(\neg B, A)$.

Proof. Towards a contradiction, suppose σ is a winning strategy for player II in that game. Consider the strategy for player I given by $\tau_1(s) = 0$ for all $s \in \mathrm{Seq}^{\{0,1,2\}}_{\mathrm{even}}$. Since $\tau_1 \otimes^{\mathrm{I}}_W \sigma \notin B$, we must have $\mathsf{Inf}(\tau_1, \sigma)$ and $h = \tau_1 \otimes^{\mathrm{II}}_W \sigma \in A$. Then, there exists k_0 such that $h(k_0) = 1$ and $\forall k' < k_0\,(h(k') = 0)$. Pick k_1 such that player II has already played (not passed) k_0+1 times after her first k_1 moves. By bounded Σ^0_1-comprehension (available in RCA_0), there exists $C = \{x : x \le k_1 \wedge \varphi(x)\}$. Define a new strategy for player I, τ_2, by putting

$$\begin{array}{ll}
\tau_2(s) = 0 & \text{if } |s| < 2\cdot k_1 \\
\tau_2(s) = 2 & \text{if } |s| = 2\cdot k_1 \\
\tau_2(s) = \begin{cases} 2 & \text{if } i \in C \\ 0 & \text{if } i \notin C \end{cases} & \text{and } |s| = 2\cdot k_1 + 2 + 2\cdot i \text{ with } i \le k_1 \\
\tau_2(s) = 0 & \text{otherwise}
\end{array}$$

In words, player I picks 0 until his opponent plays her first 1 and then he plays according to set C in his next $k_1{+}1$ moves. Then, we must have $\mathsf{Inf}(\tau_2, \sigma)$ and it is easy to check that both $B(\tau_2 \otimes^{\mathrm{I}}_W \sigma)$ and $A(\tau_2 \otimes^{\mathrm{II}}_W \sigma)$ hold, which contradicts the fact that σ is a winning strategy for player II in the game $G_W(\neg B, A)$. □

By $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}^*_W$, player II must have a winning strategy in $G_W(A, B)$, say σ_0. Given $k \in \mathbb{N}$, we will use σ_0 to decide whether or not $\varphi(k)$ holds. To that end, it will be useful to observe that

Claim 5.6.2. Let τ_1 denote the strategy for player I given by $\tau_1(s) = 0$ for all $s \in \mathrm{Seq}^{\{0,1,2\}}_{\mathrm{even}}$. Then, we must have $(\tau_1 \otimes^{\mathrm{II}}_W \sigma_0)(i) = 0$ for all $i \geq 0$.

Proof. Towards a contradiction, suppose that there is some i such that $(\tau_1 \otimes^{\mathrm{II}}_W \sigma_0)(i) = 1$ and let k_0 denote the least i satisfying that property. Since $\neg A(\tau_1 \otimes^{\mathrm{I}}_W \sigma_0)$ holds, so does $\neg B(\tau_1 \otimes^{\mathrm{II}}_W \sigma_0)$. Therefore, either $\exists i \leq k_0\, ((\tau_1 \otimes^{\mathrm{II}}_W \sigma_0)(k_0 + i + 1) = 1 \wedge \neg\varphi(i))$ holds or $\exists i \leq k_0\, (\varphi(i) \wedge (\tau_1 \otimes^{\mathrm{II}}_W \sigma_0)(k_0 + i + 1) = 0)$ holds. Now pick k_1 such that player II has played (i.e. not passed) $2{\cdot}k_0 + 2$ times after her first k_1 moves. Define a new strategy for player I, τ_2, by putting

$$\begin{aligned} \tau_2(s) &= 0 \;\text{ if }\; |s| \neq 2{\cdot}k_1 + 2 \\ \tau_2(s) &= 2 \;\text{ if }\; |s| = 2{\cdot}k_1 + 2 \end{aligned}$$

I.e., player I chooses 0 in his first k_1 moves and then he picks 1 in order to play inside set A. It is easy to check that both $A(\tau_2 \otimes^{\mathrm{I}}_W \sigma)$ and $\neg B(\tau_2 \otimes^{\mathrm{II}}_W \sigma)$ hold, a contradiction. □

It follows from Claim 5.6.2 that for each k there exists some l such that if player I plays 0 in his first l moves then player II does not pass k times and builds a sequence of k 0's in her first l moves using σ_0. More formally, $\forall k\, \exists l\, (\forall i < l(0^{(l)} \otimes^{\mathrm{II}} \sigma_0)(i) \neq 2 \;\wedge\; \mathsf{sum}(0^{(l)} \otimes^{\mathrm{II}} \sigma_0) = k)$ holds, where $\mathsf{sum}(s)$ calculates the sum of all the elements in a finite sequence. Using RCA_0, we can take a function $b : \mathbb{N} \to \mathbb{N}$ such that $b(k)$ outputs a minimal l satisfying the above property. Also, for each $k \in \mathbb{N}$ we consider a strategy for player I, τ^k, such that $\tau^k(s) = 2$ if $|s| = 2{\cdot}k$ and $\tau^k(s) = 0$ otherwise. By Δ^0_1-comprehension there exists $S \subseteq \mathrm{Seq}^{\{0,1,2\}}_{\mathrm{even}} \times \{0, 1, 2\} \times \mathbb{N}$ such that $(S)_k = \tau^k$ for each k, where $(S)_k$ denotes $\{(x, y) : (x, y, k) \in S\}$. Now observe that if player I plays using strategy $\tau^{b(k)}$ then player II must play inside set B when using the winning strategy σ_0 and so she will describe properly membership to the set $[0, z] \cap \{x : \varphi(x)\}$ for some $z \geq k$ in her answer. Thus, for each $k \in \mathbb{N}$ we have

$$\varphi(k) \leftrightarrow \exists l \left\{ \begin{array}{l} ((S)_{b(k)} \otimes^{\mathrm{II}}_W \sigma_0)(l) = 1 \wedge \forall l' < l\, ((S)_{b(k)} \otimes^{\mathrm{II}}_W \sigma_0)(l') = 0 \;\wedge \\ ((S)_{b(k)} \otimes^{\mathrm{II}}_W \sigma_0)(k + l + 1) = 1 \end{array} \right.$$

and

$$\varphi(k) \leftrightarrow \forall l \left\{ \begin{array}{l} [((S)_{b(k)} \otimes^{\mathrm{II}}_W \sigma_0)(l) = 1 \wedge \forall l' < l\, ((S)_{b(k)} \otimes^{\mathrm{II}}_W \sigma_0)(l') = 0] \;\rightarrow \\ ((S)_{b(k)} \otimes^{\mathrm{II}}_W \sigma_0)(k + l + 1) = 1. \end{array} \right.$$

Therefore the set $\{x : \varphi(x)\}$ exists by Δ^0_1-comprehension, as required. □

As an immediate consequence, we get

Theorem 5.7. *Over* RCA_0*, the following principles are pairwise equivalent*

1. ACA_0.

2. $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}^*_{L/W}$.
3. $(\Sigma^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}^*_{L/W}$.

Proof. It follows from Theorem 4.8, Lemma 5.2 and Proposition 5.6. □

6. Concluding remarks

This paper aims at a better understanding of the logical strength of Wadge and Lipschitz determinacy principles in the context of second order arithmetic. We have restricted ourselves to studying Wadge/Lipschitz determinacy, as well as the tightly related semi-linear ordering principle SLO, for the first levels of the difference hierarchy in the Cantor space. In view of the results on general determinacy in the Cantor space obtained in [11], our results on Lipschitz determinacy point out that for $\Gamma = \Delta^0_1$ or Σ^0_1, the equation Γ-$\mathsf{Det}^*_L = (\Gamma \wedge \neg\Gamma)$-$\mathsf{Det}^*$ holds (the results for Wadge determinacy are more idiosyncratic and are not that regular.) It should be noted that, however, this equivalence will cease to be true already at level Δ^0_4, since Z_2 proves Borel Lipschitz determinacy ([6]) and Z_2 does not prove general Δ^0_4-determinacy ([8]).

A number of questions have been left open throughout the text (see the remarks following Theorem 5.5):

Problem 2. Is WKL_0 equivalent over RCA_0 to Δ^0_1-SLO^*_L?

Problem 3. Is ACA_0 equivalent over RCA_0 to Σ^0_1-Det^*_W?

Problem 4. Is ACA_0 equivalent over RCA_0 to Σ^0_1-SLO^*_L?

Problem 5. Is ACA_0 equivalent over RCA_0 to Σ^0_1-SLO^*_W?

Problem 6. Does Σ^0_1-SLO^*_W imply WKL_0 over RCA_0?

In view of Theorem 5.5, a positive answer to Problem 2 would imply a positive answer to Problem 4; and a positive answer to Problem 6 would imply a positive answer to Problems 3, 4 and 5.

On top of these pending questions, two natural lines for future work suggest themselves: to extend our analysis to higher levels of the difference hierarchy in the Cantor space, as well as to stronger systems like ATR_0 and Π^1_1-CA_0, and to study the logical strength of Wadge/Lipschitz determinacy and SLO in the Baire space. In [5], a number of results in this direction have already been obtained. Most remarkably, it is shown that over RCA_0, Σ^0_1-Det_L, Δ^0_2-Det^*_L and ATR_0 are pairwise equivalent; and over ACA_0, Δ^0_1-Det_L, Δ^0_1-SLO_L and ATR_0 are pairwise equivalent. A second paper containing those and other results is in preparation.

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References

[1] A. Andretta, *Equivalence between Wadge and Lipschitz determinacy*, Annals of Pure and Applied Logic **123** (2003), 163–192.

[2] A. Andretta, *More on Wadge determinacy*, Quaderni del Dipartimento di Matematica **22** (2004), 1–26.

[3] J. Avigad, K. Simic, *Fundamental notions of analysis in subsystems of second-order arithmetic.* Annals of Pure and Applied Logic 139 (2006) 138–184.

[4] W. Chan, *A Lipschitz determinacy principle equivalent to Weak Köning Lemma*, preprint, 2021.

[5] M.J.S. Loureiro, *Semilinear order property and infinite games*, Ph.D. thesis, University of Seville, Spain, 2016.

[6] A. Louveau and J. Saint Raymond, *Borel classes and closed games: Wadge-type and Hurewitcz-type results*, Transactions of the American Mathematical Society **304** (1987), 431–467.

[7] D. Martin, Σ^0_4*–Determinacy*, circulated handwritten notes, dated April 24, 1974.

[8] A. Montalbán and R. A. Shore, *The limits of determinacy in second order arithmetic*, Proceedings of the London Mathematical Society **104** (2012), 223–252.

[9] T. Nemoto, *Determinacy of Wadge classes and subsystems of second order arithmetic*, Math. Log. Quart. **55** (2009), 154–176.

[10] T. Nemoto, *Determinacy of Infinite Games and Reverse Mathematics*, Ph.D. thesis, Tohoku University, Sendai, Japan, 2009.

[11] T. Nemoto, M. O. MedSalem, and K. Tanaka, *Infinite games in the Cantor space and subsystems of second order arithmetic*, Math. Log. Quart. **53** (2007), 226–236.

[12] P. Shafer, *Notes on finding a point on the boundary of a closed-but-not-open set.* 2019, personal communication.

[13] S. G. Simpson, *Subsystems of Second Order Arithmetic*, Springer-Verlag, 1999.

[14] J. R. Steel, *Determinateness and subsystems of analysis*, Ph.D. thesis, University of California, Berkeley, 1977.

[15] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis I:* Δ^0_2*-games*, Zeitschrift f. math. logic und Grundlagen d. Math. **36** (1990), 481–491.

[16] K. Tanaka, *Weak axioms of determinacy and subsystems of analysis II:* Σ^0_2*- games*, Ann. Pure Appl. Logic **52** (1991), 181–193.

[17] W. W. Wadge, *Reducibility and Determinateness on the Baire Space*, Ph.D. thesis, University of California, Berkeley, 1983.

Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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